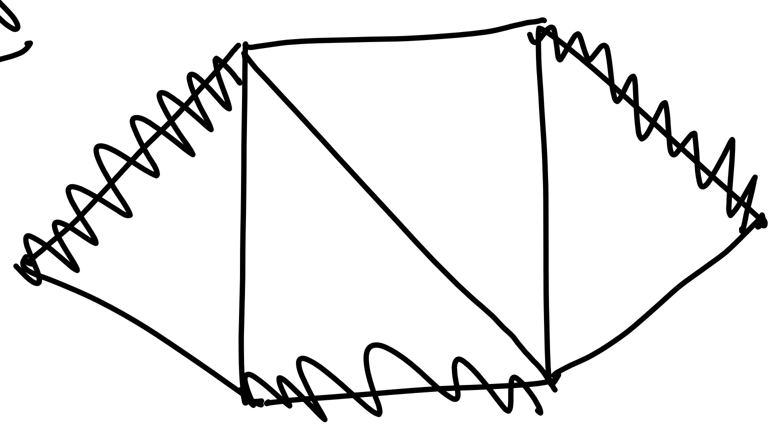
11/7/2007

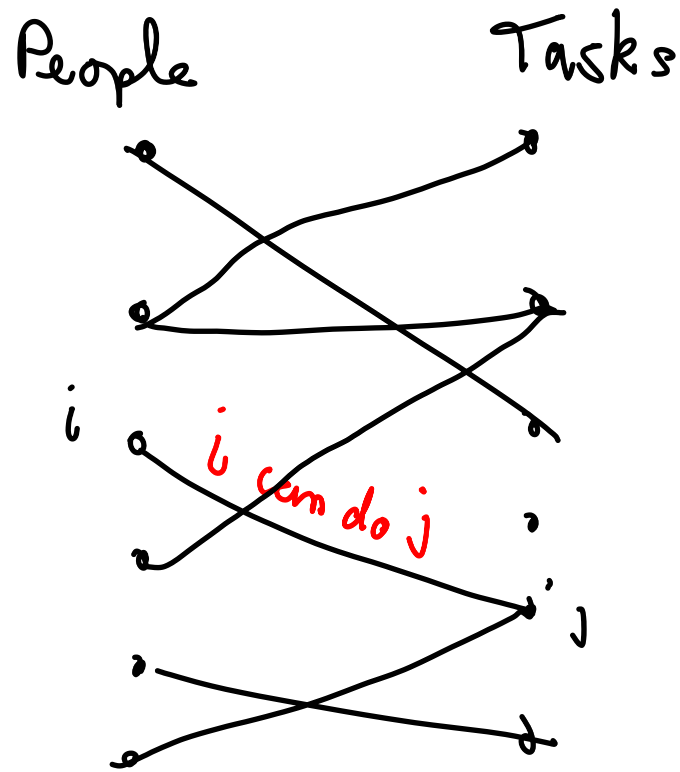
Matchings



Matching = Set of pairwise disjoint edges

Perfect if every vertex is on an edge of Matching.

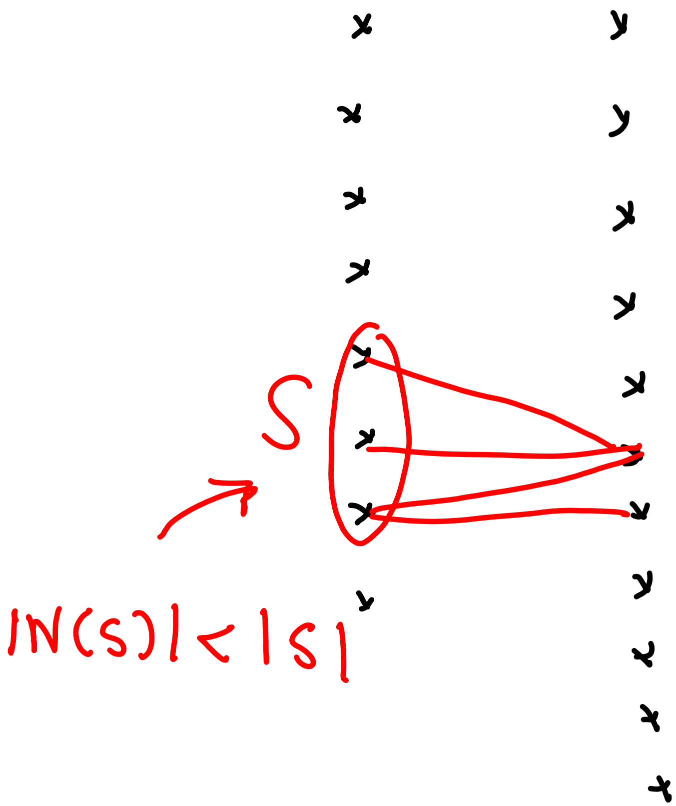
Bipartite Case



people = # tasks

A perfect matching assigns each task to one person

Hall's Theorem



$|A| \leq |B|$

When can we find a matching of size $|A|$.

If $\exists S \subseteq A$ s.t. $|N(S)| < |S|$ then $\nexists$ matching of size $|A|$.

Hall: If $|N(S)| \geq |S|$, $\forall S \subseteq A$ then $\exists$ matching of size $|A|$.

Assume Hall's condition

$$P = A \cup B$$

and $a \le b$ iff (a, b) is an edge.

[+ $a \le a$ etc.]

$$S = \{ a_1, a_2, \dots, a_h, b_1, b_2, \dots b_k \} \quad s = h + k$$

is largest anti-chain in P

$$N(a_1, a_2, \dots a_h) \subseteq B \setminus \{ b_1, b_2, \dots, b_k \}$$

a_i — b_j not allowed, else $a_i \le b_j$

Hall Condition: $h \le |B| - k$ or $\underline{\underline{|B| \ge s}}.$

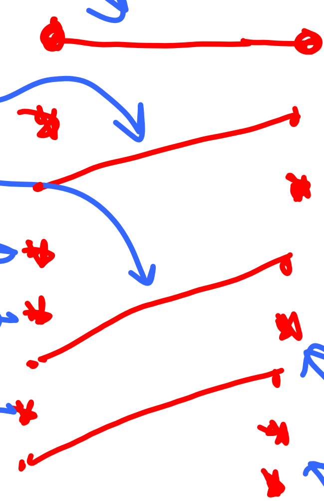
Apply Dilworth theorem:

$P = \bigcup$ of s chains.

matching of size m

\# chains

$m + (|A| - m) + (|B| - m) = s \le |B|$



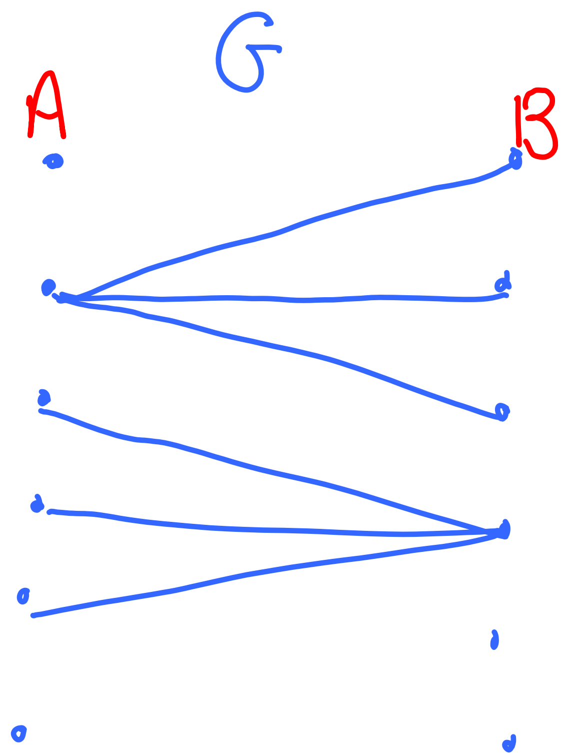
set of chains

2 chains

$$m + |A| - m + |B| - m \le |B|$$

or

$$m \geq |A|.$$



Every vertex has same degree $k \geq 1$

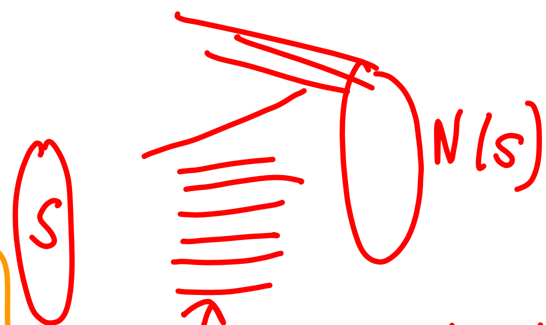
Marriage Theorem: G has a perfect matching

$|A| = |B|$: #edges in G $= k|A| = k|B|$.

Halls Condition:

$\Downarrow$

$\boxed{G = \bigcup k \text{ disjoint perfect matchings}}$



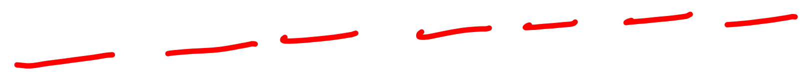
$k|S| \leq k\,|N(S)|$

Intervals on real line

$mn+1$ intervals $I_j = [a_j, b_j]$, $1 \le j \le mn+1$.

Thm

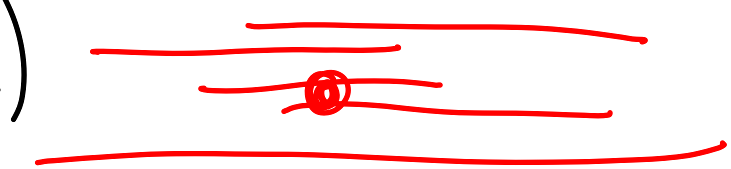
Either



(i) $\exists$ $m+1$ disjoint intervals or

(ii) $\exists$ $n+1$ intervals with a common point

(or both)



$$P = \{I_1, I_2, \ldots, I_M\} \qquad M = mn+1.$$

$$I_j < I_k \iff b_j < a_k$$

Suppose there is no chain of length $m+1$.

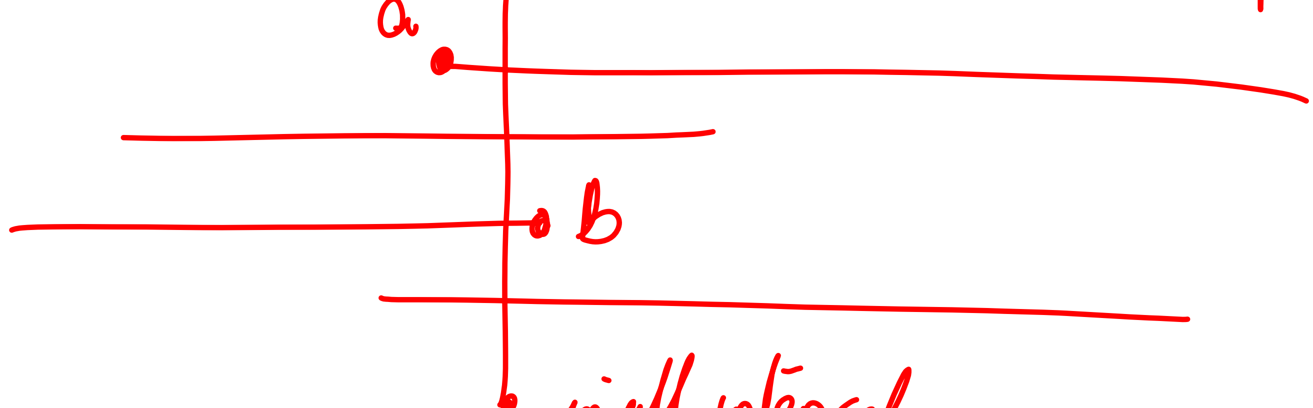
$\Rightarrow$ we need $\left\lceil \dfrac{mn+1}{m} \right\rceil$ chains to cover P.

$\geq n+1$

Dilworth Thm $\Longrightarrow \exists$

anti-chain $\underbrace{I_{j_1}, I_{j_2}, \dots I_{j_{n+1}}}_{}$

Must have a common point.



in all intervals.