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Chains: $a_1 < a_2 < \cdots < a_p$

Anti-Chains: $b_1, b_2, \ldots b_q$ are pair-wise incomparable.

C is a chain
A is an anti-chain $\Rightarrow |C \cap A| \leq 1$

Covers by chains & anti-chains.

Covers by anti-chains

Suppose $A_1, A_2, \ldots, A_m$ are anti-chains

$C = \{a_1 < a_2 < \cdots < a_p\}$ is

a chain.

Then $m \geq p$

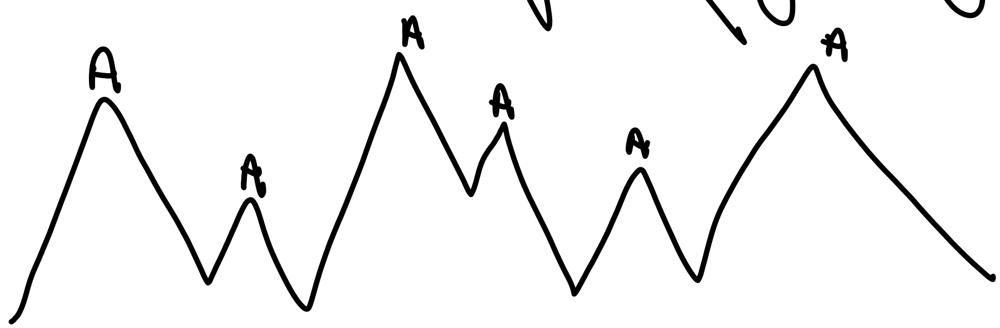
Smallest m = largest p

Proof: by induction on length of longest chain $- \mu$

Base Case : $\mu = 1 \Rightarrow P$ is an anti-chain.

$A = \{\text{maximal elements}\}$

x is maximal if $\nexists y : y > x$



$P' = P / A$: P' does not have a chain of length μ ✓

$P' = \bigcup_{i=1}^{\mu-1} A_i \leftarrow$ anti-chain

$P = A_1 \cup \cdots \cup A_{\mu-1} \cup A$.

$a_1 < a_2 < \cdots < a_\mu < y \in A$ ($\notin A$)

Contradiction.

Covers by chains

Suppose $C_1, C_2, \ldots, C_m$ are chains

$A = \{a_1, a_2, \cdots a_p\}$ is

an anti-chain.

Then $m \geq p$

Smallest m = largest p Dilworth's Theorem

Proof: by induction on $|P|$.

Base Case: $|P| = 0$ ✓

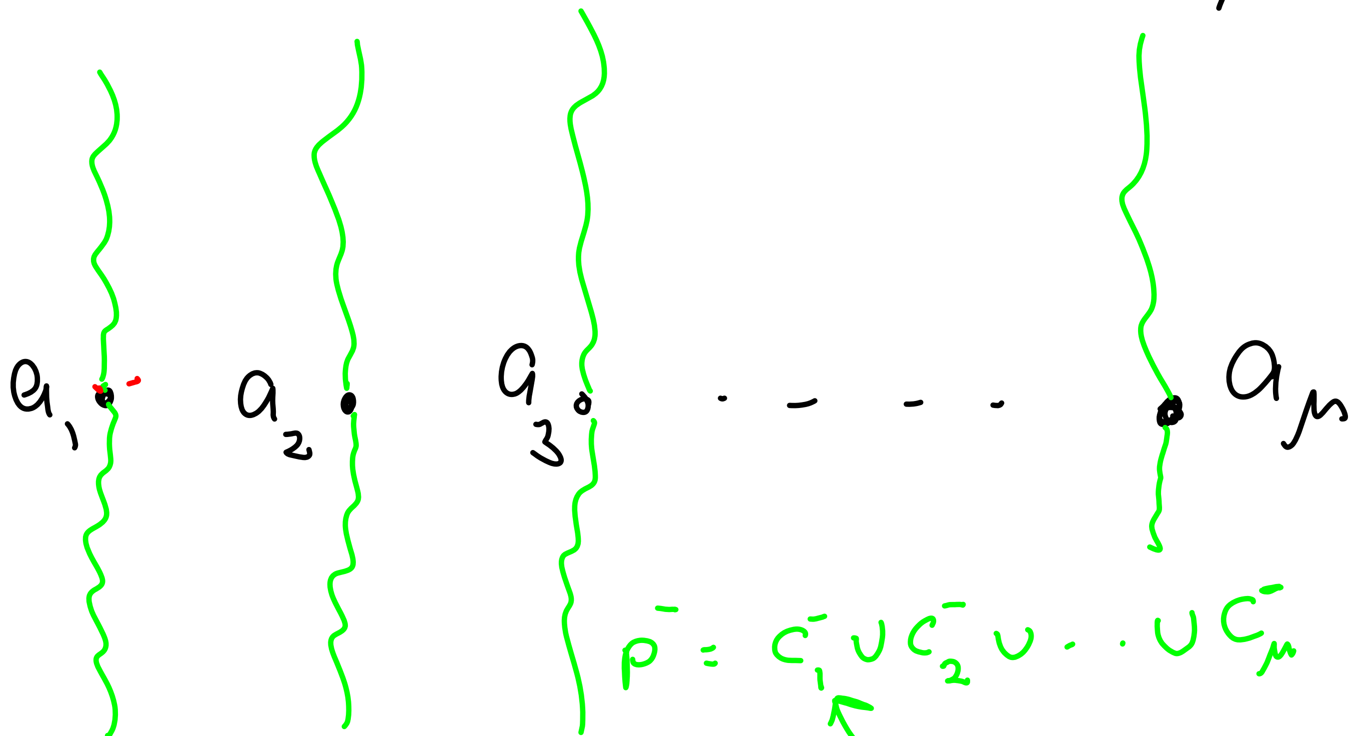
Assume $|P| > 0$

μ is the size of a largest anti-chain.

$C = x_1 < x_2 < \cdots < x_p$
is a longest chain.

Case 1: Every anti-chain in $P \setminus C$ has $\leq \mu - 1$ elements. — we use induction to find $C_1, C_2, \ldots C_{\mu-1}$ & add C.

Case 2: $\exists$ anti-chain $A = \{a_1, a_2, \dots, a_m\} \subseteq P \setminus C$



$P^- = C_1^- \cup C_2^- \cup \dots \cup C_m^-$

(i) $P = P^+ \cup P^-$

(ii) $P^+ \cap P^- = A$

(iii) $|P^-| < |P|$

$x_p \notin P^-$

$P^+ = \{ x \in P : x \geq \text{some } a_i \}$

$P^- = \{ x \in P : x \leq \text{some } a_i \}$

Proof of Erdős - Szekeres

$a_1, a_2, \dots, a_{n^2+1}$ contains a monotone sequence of length $n+1$.

$P = \{ (i, a_i) : 1 \leq i \leq n^2+1 \}$

$(i, a_i) < (j, a_j)$ if $\begin{array}{l} i < j \\ a_i \leq a_j \end{array}$

Suppose P has no chain of length $n+1$

$\Rightarrow$ we need $\geq n+1$ chains to cover P

So there is an anti-chain A of size $n+1$.

$$(i_1, a_{i_1}) \quad (i_2, a_{i_2}) \quad \ldots \quad (i_{n+1}, a_{i_{n+1}})$$

ordered so that $i_1 < i_2 < \cdots < i_{n+1}$

$$\Rightarrow a_{i_t} > a_{i_{t+1}}$$

else A is not an anti-chain.