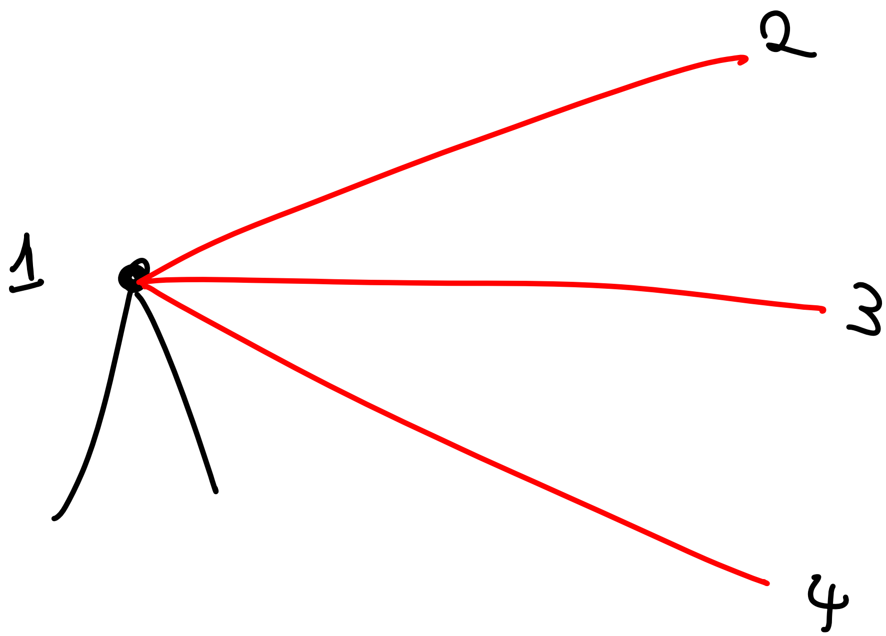
10\26\07

Ramsey Theory



smallest value

Generalisation: Fix k, l ; $\exists\, R(k,l)$ such that if $n \geq R(k,l)$ and you 2-color edges of K_n, then Red K_k or Blue K_l.

$$R(3,3) = 6$$

$$R(1,k) = R(k,1) = 1$$

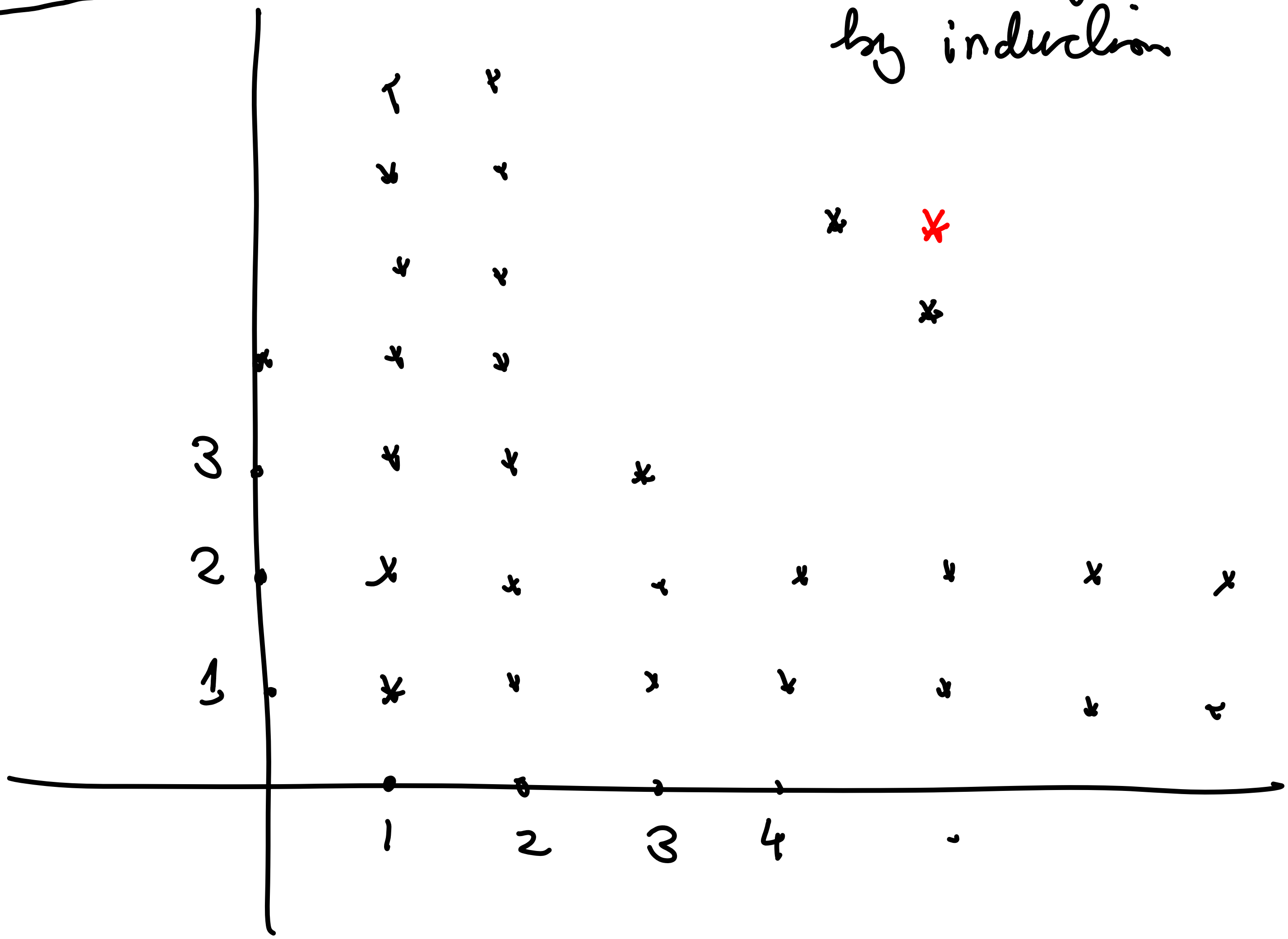
$$R(2,k) = R(k,2) = k$$

$$R(4,4) = 18$$ Hard

$$R(5,5) = ??$$

* exists

* exists everywhere,
by induction

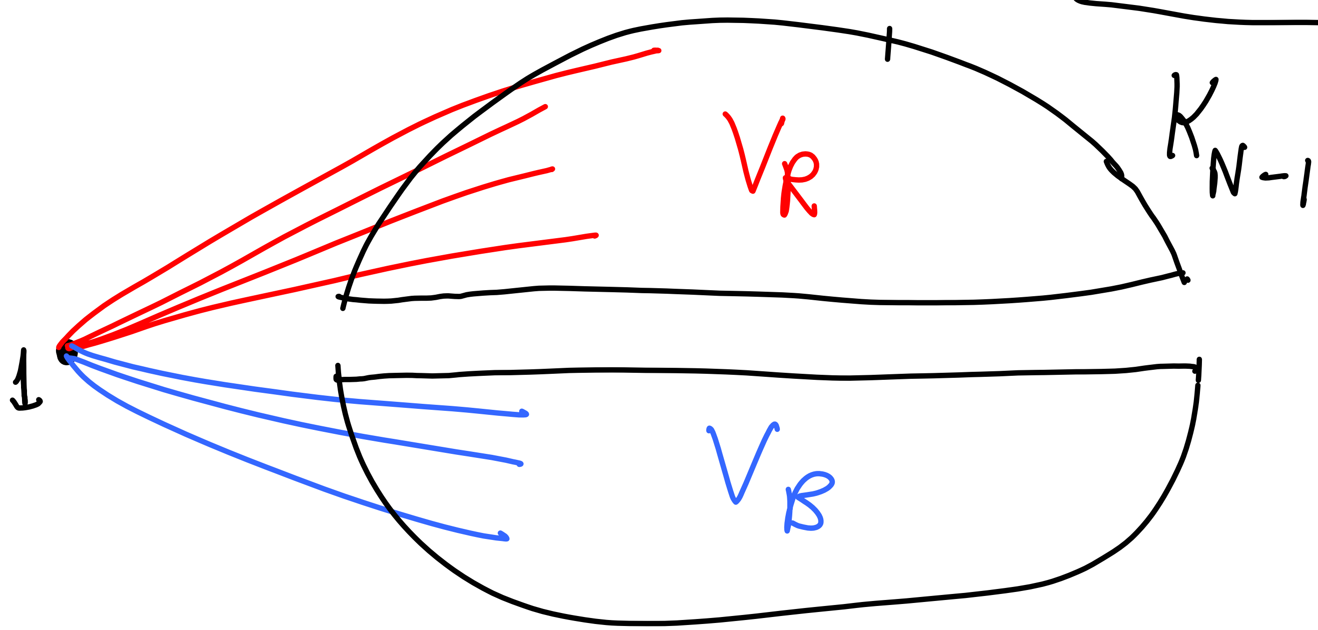


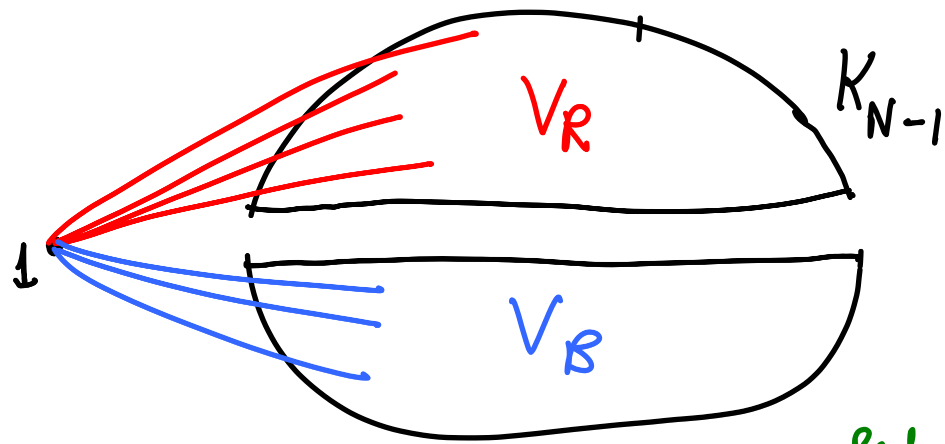
$$R(k,l) \le R(k,l-1) + R(k-1,l)$$

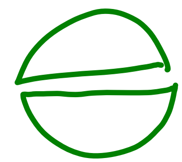
Induction proof of existence. Assume RHS exists.

$$N = R(k,l-1) + R(k-1,l)$$

$$\boxed{\begin{array}{c} |V_R| \geq R(k-1,l) \\ \text{or} \\ |V_B| \geq R(k,l-1) \end{array}}$$





$|V_R| \geq R(k-1, l)$: Inside  $\exists$ Red K_{k-1} or Blue K_l

Add vertex 1 to get red K_k.

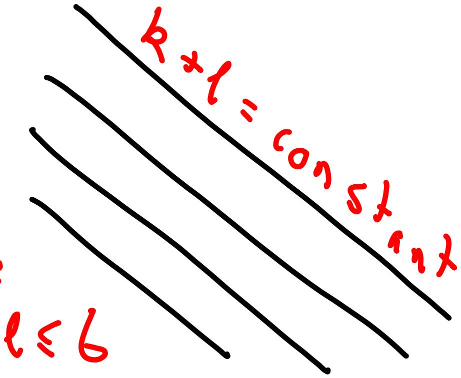
or

$|V_B| \geq R(k, l-1)$

$$R(k,l) \leq \binom{k+l-2}{k-1}$$

Induction on $k+l$.

Base Case:
Check: $k+l \leq 6$



$$R(k,l) \leq R(k,l-1) + R(k-1,l) \quad \text{just proved}$$

$$\leq \binom{k+l-3}{k-1} + \binom{k+l-3}{k-2} \quad \text{Induction}$$

$$= \binom{k+l-2}{k-1} \qquad \square$$

$$R(k,k) \leq \binom{2k-2}{k-1} < 4^k.$$

$$R(k,k) > 2^{k/2}$$

We show that if $n \le 2^{k/2}$ then

$\exists$ a way of 2-coloring K_n, so that

$\nexists$ a Red k-clique and $\nexists$ Blue k-clique.

We use the probabilistic method.

Randomly color edges.

$$\mathcal{E}_R = \{\exists \text{ red } k\text{-clique}\}$$

$$\mathcal{E}_B = \{\exists \text{ blue } k\text{-clique}\}$$

$$\Pr(\mathcal{E}_R \cup \mathcal{E}_B) < 1$$

$C_1, C_2, \ldots, C_{\binom{n}{k}}$ are the k-cliques of K_n.

$$\mathcal{E}_{R,i} := \{C_i \text{ is Red}\}$$

$$\Pr(\mathcal{E}_R \cup \mathcal{E}_B) \le \Pr(\mathcal{E}_R) + \Pr(\mathcal{E}_B)$$

$$= 2\Pr(\mathcal{E}_R)$$

$$= 2\Pr\left(\bigcup_{i=1}^{N} \mathcal{E}_{R,i}\right)$$

$$\le 2\sum_{i=1}^{N} \Pr(\mathcal{E}_{R,i})$$

$$= 2N\left(\frac{1}{2}\right)^{\binom{k}{2}}$$

$$\le 2\frac{n^k}{k!}\left(\frac{1}{2}\right)^{\binom{k}{2}} \le 2\frac{2^{k^2/2}}{k!}\left(\frac{1}{2}\right)^{\binom{k}{2}} = \frac{2^{1+k/2}}{k!} \underset{k\ge 4}{<} 1$$