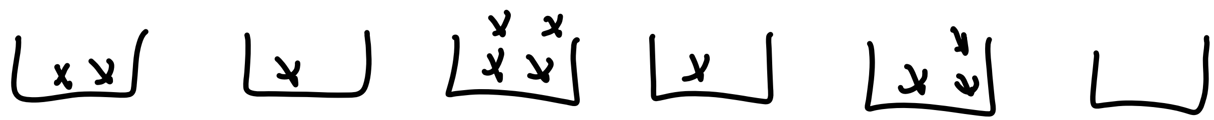
10/26/2007

Pigeon Hole Principle

m objects



n boxes

At least one box gets $\geq$ average $\frac{m}{n}$

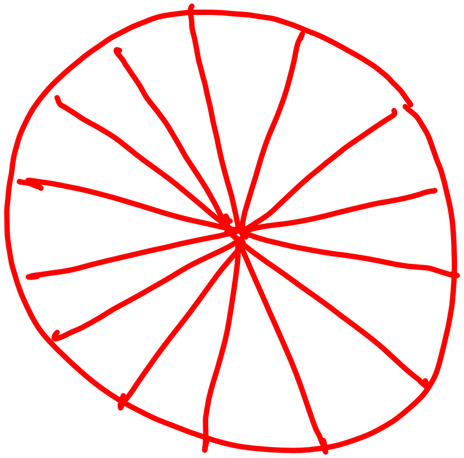
$\Rightarrow \geq \left\lceil \frac{m}{n} \right\rceil$.

$A \subseteq [2n] \qquad |A| = n+1$

$\Rightarrow \exists\ x, y \in A$ s.t. x & y are co-prime.

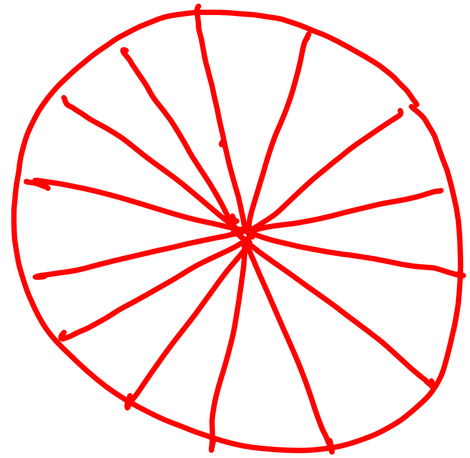
a & $a+1$ are always co-prime

Disk 1



100 Red colored section
100 Blue colored section

Disk 2



Arbitrarily colored red and blue.

Place disk 2 onto disk 1 so that section line up.

q_i = # of color matches.

Claim: $\exists\, i$ s.t. $q_i \geq 100$.

Claim: $q_1 + q_2 + \cdots + q_{200} = 200 \times 100$

$\Rightarrow \exists i : q_i \geq 100.$

Double Counting

Explanation:

$$\begin{array}{cc} & \text{sector } j \text{ of disk 2} \\ \text{position } i & \left[\begin{array}{c} 1 \end{array}\right] \end{array}$$

j sits on sector of same color.

#1's $= q_i$

#1's $= 100$

Alternatively:

Place Disk 2 randomly on top of Disk 1.

$$X_j = \begin{cases} 1 & \text{sector } j \text{ sits on same color} \\ 0 & \text{otherwise} \end{cases}$$

$$E(X_j) = \frac{1}{2}$$

$$X = X_1 + \cdots + X_{200} = \#\text{matches}$$

$$E(X) = 100 \Rightarrow \exists \text{ a way of getting } 100.$$

Erdős - Szekeres

$a_1, a_2, \ldots, a_{k^2+1}$ is a sequence of integer.

$\exists$ a monotone subsequence of length $k+1$.

$$a_{i_1} \leq a_{i_2} \leq \cdots \leq a_{i_{k+1}}$$

or

$$a_{i_1} \geq a_{i_2} \geq \cdots \geq a_{i_{k+1}}$$

$4, 8, 3, 7, 2$ $\qquad k=2$

$*$ $\quad$ $*$ $\quad$ $*$

a_i : $\ell(a_i)$ = length of longest monotone increasing sequence starting with a_i.

$$a_i \leq a_i^1 \leq \cdots \leq a_i^{\ell-1}$$

$$\times \quad \times \quad \times \quad a_i \times \times \quad a_i^1 \times \times \quad \times \times a_i^2 \times \times \cdots \cdots a_i^{\ell-1} \times \times$$

Case 1 : $\exists\, i$: $\ell(a_i) \geq k+1$ — done

Case 2 : $\ell(a_i) \leq k$, $i = 1, 2, \ldots, k^2+1$

a_i : $\ell(a_i) = i$

k^2+1 objects $\longrightarrow$ k buckets

$\exists$ bucket with $\geq k+1$ numbers

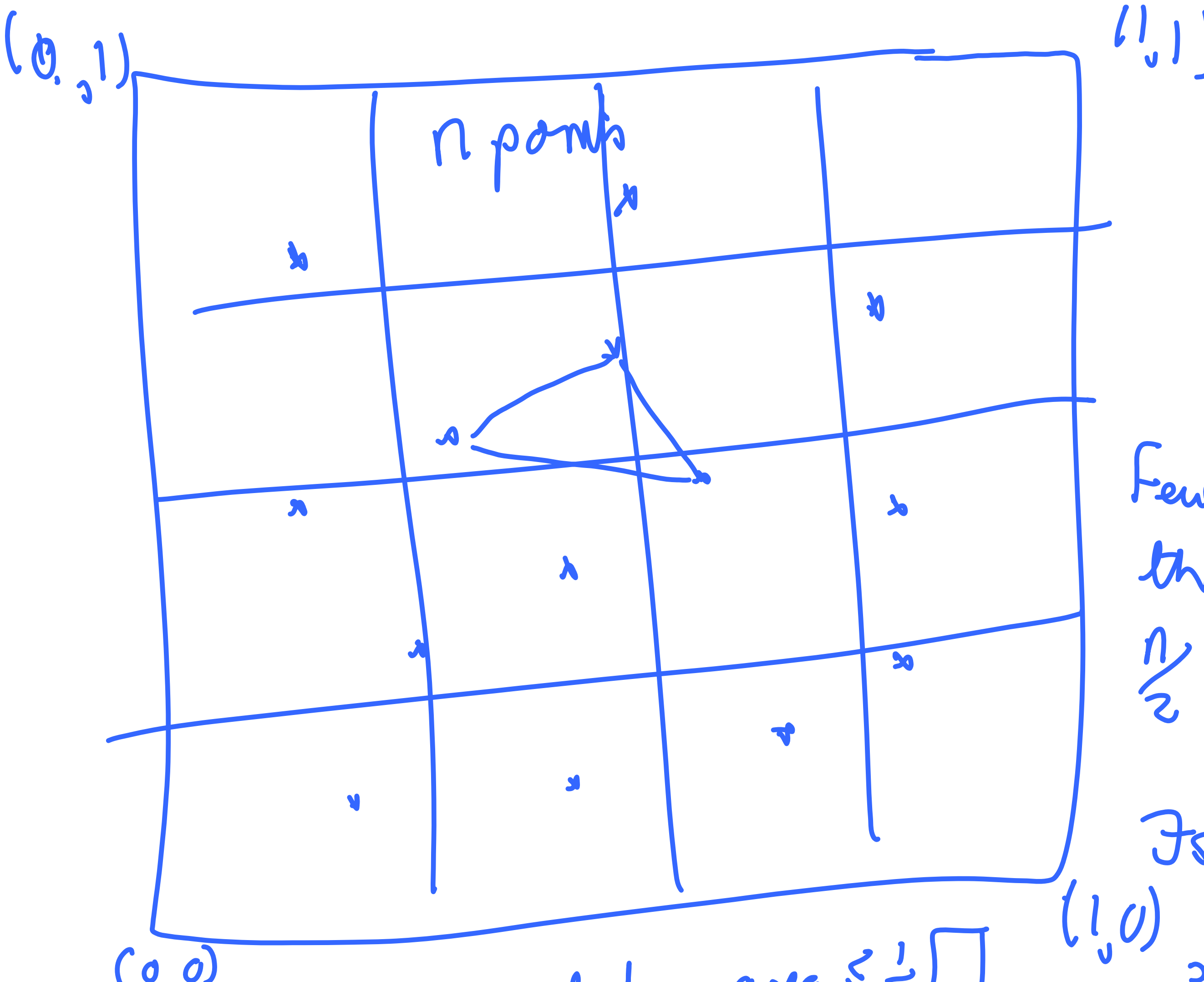
bucket t

$$\boxed{\begin{matrix} a_{i_1} & a_{i_2} & a_{i_3} \\ a_{i_4} & \cdots & a_{i_{k+1}} \\ \cdots & & \end{matrix}}$$

$i_1 < i_2 < \cdots < i_{k+1}$

Claim: $a_{i_1} \geq a_{i_2} \geq \cdots \geq a_{i_{k+1}}$

$a_{i_r} < a_{i_{r+1}} \leq a'_{i_{r+1}} \leq \cdots \leq a^{t-1}_{i_{r+1}}$



Fewer than $\frac{n}{2}$ square $\Downarrow$ $\exists$ square with 3 pts

3 pts have area $\leq \frac{1}{2}\square$