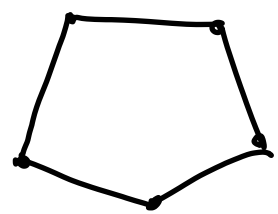
10\17\2007

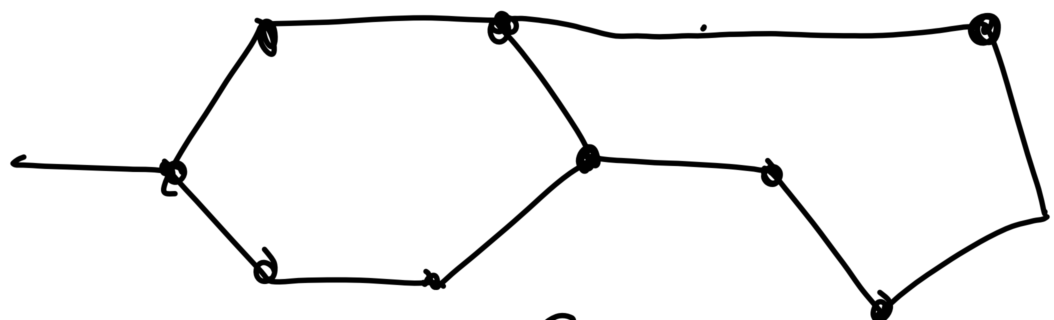
G is a graph.

A cycle



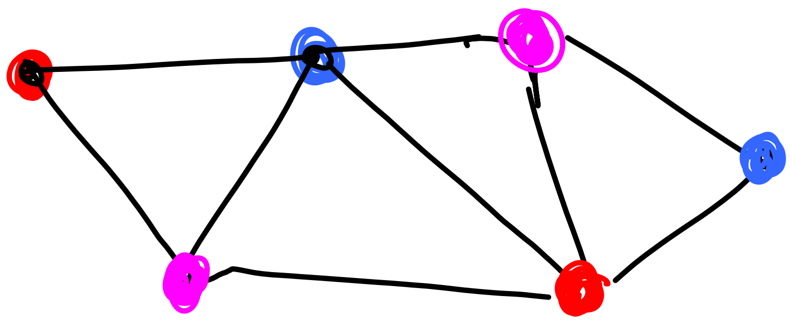
Closed sequence of edges.

Girth := length of shortest cycle in G



Girth = 5

Chromatic Number



$\chi(G) = 3$

Minimum number of colors needed to color each vertex so that each edge joins vertices of a different color.

= minimum k s.t.

$V = V_1 \cup V_2 \cup \dots \cup V_k$ where each V_i is independent

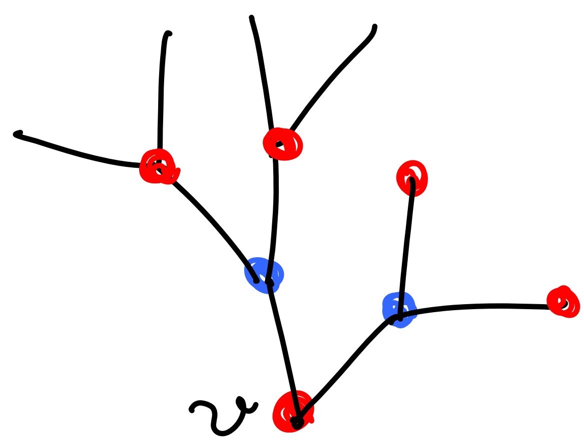
$$\chi(G) \geq \frac{n}{\alpha(G)}.$$

Is it true that

$$\chi(G) \leq f[\text{girth}] \; ? \quad (*)$$

for some function f.

Large girth:



$(*)$ is not true.

Theorem

Let $a, b > 0$ be positive integers. Then there exists a graph with girth $g \geq a$ and chromatic number $\chi \geq b$.

Proof: probabilistic method.

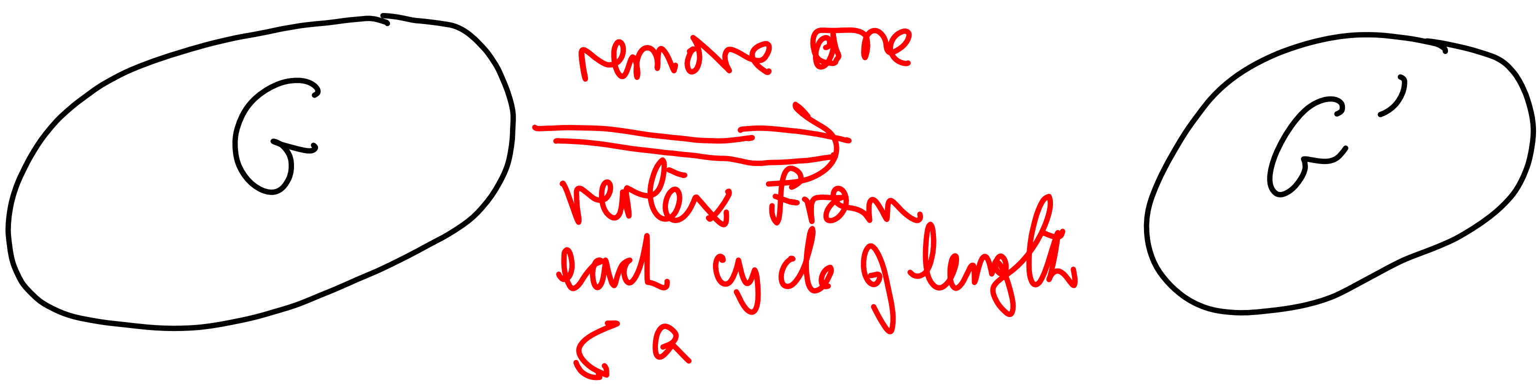
Choose a graph at random

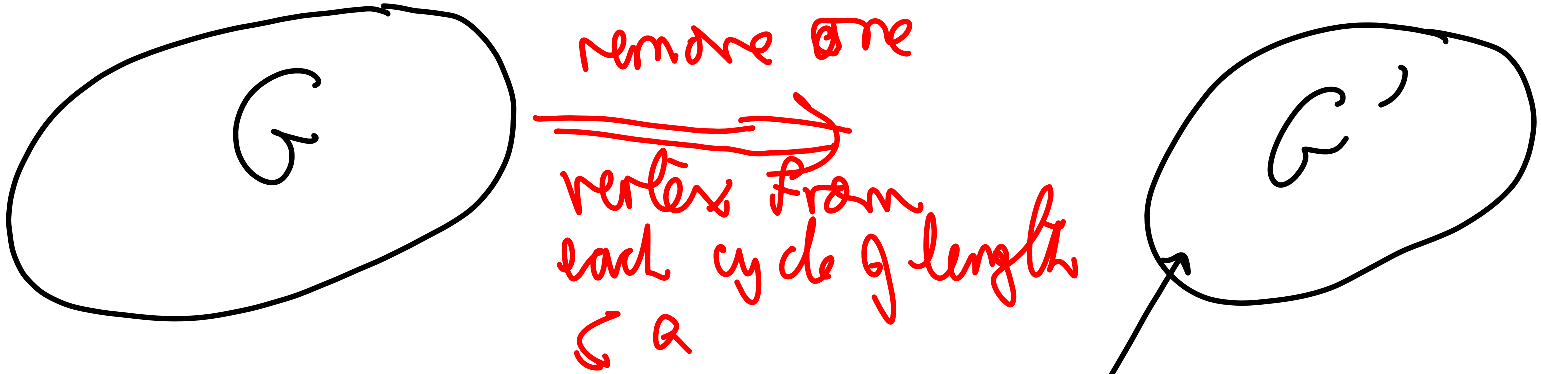
Lemma 1

Let d be such that $\frac{d}{3\log d} \geq b$. Then there exists a graph G with $n \geq 100 d^a$ vertices and such that

(i) At most $2d^a$ cycles of length $\leq a$

(ii) No independent set of size $\geq \frac{2\log d}{d} n$.



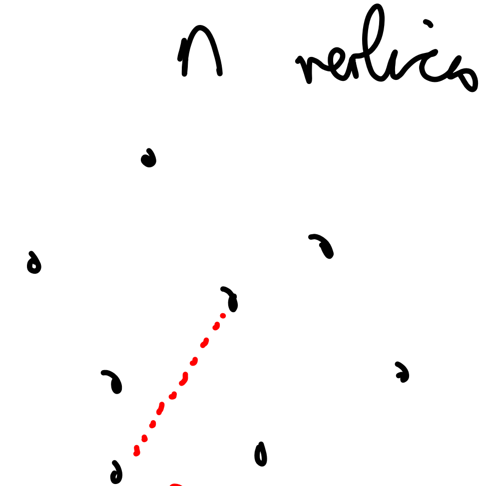


Girth $> g$

$$\chi \geq \frac{.98\,n}{\frac{2\log d}{d}\,n} > \frac{d}{3\log d} > k.$$

Proof of Lemma

Random graph $G_{n,p}$:



each edge is included with probability p, independently.

$$G = G_{n,p} \text{ where } p = \frac{d}{n}.$$

Show that with positive prob., G has both prop.

Simple Inequality:

Markov's Inequality.

If X is a non-negative random variable then

$$P(X \geq t) \leq \frac{E(X)}{t}, \quad \forall t > 0.$$

$$E(X) = E(X \mid X \geq t) P(X \geq t) + E(X \mid X < t) P(X < t)$$

$$\geq t P(X \geq t).$$

$X = \#$ of cycles of length $\le a$.

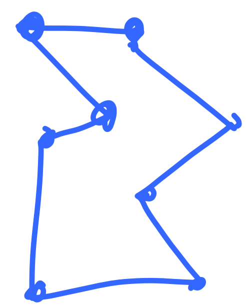
$$E(X) \le \sum_{k=3}^{a} E[\#\text{ cycles of length } k]$$

$\#$ of cycles of length k = $X_1 + X_2 + \cdots$

Enumerate possible cycles 1, 2, . . .

where $X_i = \begin{cases} 1 & \text{cycle } i \text{ exists} \\ 0 & \text{otherwise} \end{cases}$

$\Pr(X_i = 1) = p^k$



← k edges

$$\boxed{E(X) \le \sum_{k=3}^{a} p^k \cdot n^k < d^a}$$

$\ge \#$ cycles of length k in K_n.

$$E(X) < d^{a}$$

$$\Pr\left(X \geq 2d^{a}\right) < \frac{1}{2}.$$

Independence No.:

$$\Pr\left(\alpha(G) \geq \frac{2\log d}{d} n\right)$$