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Extremal Set Theory

P is some property

$$\mathcal{A} \subseteq \mathcal{P}([n])$$

↑
collection
of subsets
of $[n]$

power set:
all 2^n subsets

General Question: If $\mathcal{A}$ has property P ,
what can we say about $|\mathcal{A}|$?

Sperner Families:

$\mathcal{A}$ is a Sperner family if

$$A_1, A_2 \in \mathcal{A} \Rightarrow A_1 \not\subseteq A_2 \\ \text{ \& } A_2 \not\subseteq A_1$$

For k

$$\mathcal{A}_k = \{ A \subseteq [n] : |A| = k \} \quad |\mathcal{A}_k| = \binom{n}{k}$$

Largest $\mathcal{A}_k$ is $\mathcal{A}_{\lfloor n/2 \rfloor}$, $|\mathcal{A}_{\lfloor n/2 \rfloor}| = \binom{n}{\lfloor n/2 \rfloor}$

We show

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq 1$$

LYM inequality

$$\binom{n}{|A|} \leq \binom{n}{\lfloor n/2 \rfloor}$$

$$1 \geq \sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{\lfloor n/2 \rfloor}} = \frac{|\mathcal{A}|}{\binom{n}{\lfloor n/2 \rfloor}}$$

Proof of LYM

Let π be a random permutation of $[n]$.

$$|A|=a \quad \mathcal{E}_A = \left\{ \pi : A = \{ \pi(1), \pi(2), \dots, \pi(a) \} \right\}$$

$$\begin{array}{c} \mathcal{E}_A \sim \underbrace{\pi(1) \ \pi(2) \ \dots \ \pi(a)}_A \quad \pi(a+1) \ \dots \ \pi(b) \quad \pi(n) \\ \mathcal{E}_B \sim \underbrace{\hspace{10em}}_B \end{array}$$

BOTH CANNOT HAPPEN

If $A, B \subseteq \mathcal{A}$

$\mathcal{E}_A$ & $\mathcal{E}_B$ are disjoint: suppose $|B| \geq |A|$

$$\mathcal{E}_{\{5,9,12\}} \begin{array}{cccc} 5 & 9 & 12 & \dots \\ 5 & 12 & 9 & \dots \\ 9 & 5 & 12 & \dots \\ \vdots & & & \end{array}$$

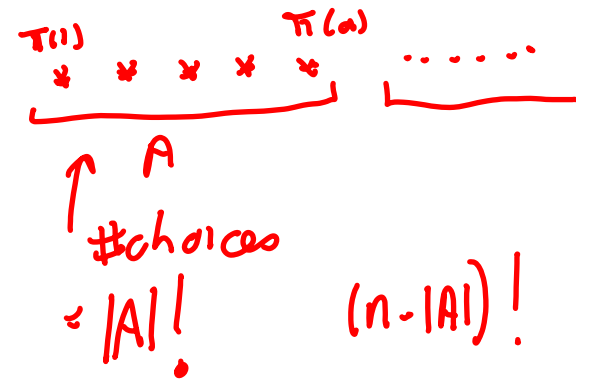
$$\underline{1} \geq P_r \left(\bigcup_{A \in \mathcal{A}} E_A \right)$$

$$= \sum_{A \in \mathcal{A}} P_r(\mathcal{E}_A)$$

events are disjoint

$$= \sum_{A \in \mathcal{A}} \frac{|A|! (n-|A|)!}{n!} = \sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}}$$

$$P_1(E_A) = \frac{\# \pi : E_A \text{ occurs}}{n!}$$



Kraft's Inequality

$x_1, x_2, \dots, x_m$ are sequences over alphabet

Σ .

x_i has length n_i

$|\Sigma| = r$. $n = \max\{n_1, \dots, n_m\}$

$$\sum_{i=1}^m r^{-n_i} \leq 1$$

Assume

~~$\nexists i \neq j$~~ s.t. x_i is a pre-fix of x_j

NOT
ALLOWED

$x_i = a_1 a_2 a_3 \dots a_{n_i}$

$n_i \leq n_j$

$x_j = a_1 a_2 a_3 \dots$

$a_{n_i} b_{n_i+1} \dots b_{n_j}$

$Z = z_1 z_2 \dots z_n$ is a random sequence over Σ

$$E_i = \{x_i = z_1 z_2 \dots z_{n_i}\}$$

E_i & E_j are disjoint events

$$E_i \quad x_i = z_1 z_2 \dots z_{n_i}$$

$$E_j \quad x_j = z_1 z_2 \dots z_{n_j}$$

BOTH CAN'T
HAPPEN

$$\sum_{i=1}^m P_i(E_i) \leq 1.$$

$\nwarrow$
 $\frac{1}{|\Sigma|^{n_i}}$