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Extremal Set Theory

P is some property

$\mathcal{A} \subseteq \mathcal{P}([n])$

collection of subsets of $[n]$

power set: all 2^n subsets

General Question: If $\mathcal{A}$ has property P, what can we say about $|\mathcal{A}|$

Sperner Families:

$\mathcal{A}$ is a Sperner family if

$$A_1, A_2 \in \mathcal{A} \Rightarrow A_1 \not\subseteq A_2$$

$$\& \; A_2 \not\subseteq A_1$$

Fix k

$$\mathcal{A}_k = \{ A \subseteq [n] : |A| = k \} \qquad |\mathcal{A}_k| = \binom{n}{k}$$

Largest $\mathcal{A}_k$ is $\mathcal{A}_{\lfloor n/2 \rfloor}$, $|\mathcal{A}_{\lfloor n/2 \rfloor}| = \binom{n}{\lfloor n/2 \rfloor}$

We show

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq 1$$

LYM inequality

$$\binom{n}{|A|} \leq \binom{n}{\lfloor n/2 \rfloor}$$

$$1 \geq \sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{\lfloor n/2 \rfloor}} = \frac{\mathcal{A}}{\binom{n}{\lfloor n/2 \rfloor}}$$

Proof of LYM

Let π be a random permutation of $[n]$.

$|A| = a$ $\quad \mathcal{E}_A = \{ \pi : A = \{\pi(1), \pi(2), \ldots, \pi(a)\} \}$

$\underbrace{\pi(1) \quad \pi(2) \ldots \pi(a)}_{A} \quad \pi(a+1) \ldots \pi(b) \quad \pi(n)$

$\mathcal{E}_A$ & $\mathcal{E}_B$ BOTH CANNOT HAPPEN

B

$\mathcal{E}_{\{5,9,12\}}$

5 9 12 ...
5 12 9 ...
9 5 12 ...
⋮

If $A, B \in \mathcal{A}$
$\mathcal{E}_A$ & $\mathcal{E}_B$ are disjoint: suppose $|B| \geq |A|$)

$$1 \geq \Pr\Big(\bigcup_{A \in \mathcal{A}} \mathcal{E}_A\Big)$$

$$= \sum_{A \in \mathcal{A}} \Pr(\mathcal{E}_A)$$

events are disjoint

$$= \sum_{A \in \mathcal{A}} \frac{|A|!\,(n-|A|)!}{n!} = \sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}}.$$

$$\underbrace{\overset{\pi(1)}{*}\quad *\quad *\quad *\quad \overset{\pi(a)}{*}}_{A}\qquad \underbrace{\cdots\cdots}$$

#choices $= |A|!$

$(n-|A|)!$

$$\Pr(\mathcal{E}_A) = \frac{\#\pi : \mathcal{E}_A \text{ occurs}}{n!}$$

Kraft's Inequality

$x_1, x_2, \ldots, x_m$ are sequence over alphabet

Σ.

x_i has length n_i

$|\Sigma| = r$. $\underline{n = \max\{n_1, \ldots n_m\}}$

$$\boxed{\sum_{i=1}^{m} r^{-n_i} \le 1}$$

Assume

$\nexists\, i \neq j$ s.t. x_i is a pre-fix of x_j

NOT ALLOWED

$x_i = a_1 a_2 a_3 \cdot \cdot \cdot \cdot \, a_{n_i}$ $\quad n_i \le n_j$

$x_j = a_1 a_2 a_3 \cdot \quad a_{n_i} b_{n_i+1} \cdots b_{n_j}$

$Z = z_1 z_2 \cdots z_n$ is a random sequence over Σ

$\mathcal{E}_i = \{ x_i = z_1 z_2 \cdots z_{n_i} \}$

$\mathcal{E}_i$ & $\mathcal{E}_j$ are disjoint events

$\mathcal{E}_i \quad x_i = z_1 z_2 \cdots z_{n_i}$

$\mathcal{E}_j \quad x_j = z_1 z_2 \cdots \cdots z_{n_j}$

BOTH CAN'T HAPPEN

$$\sum_{i=1}^{m} \Pr(\underbrace{\mathcal{E}_i}_{\frac{1}{r^{n_i}}}) \leq 1.$$