10/10/2007

Theorem

If X, Y are random variables on a probability space Ω then

$$E(X+Y) = E(X) + E(Y)$$

$(X+Y)(\omega) = X(\omega) + Y(\omega)$

No assumptions about independence is needed.

(unlike $E(XY) = E(X)E(Y)$ requires indep.)

Binomial Distribution

$B(n,p) = \#$ of head ($Pr(H) = p$)
when you toss n coins.

$B(n,p)(HHTHTTH) = 4$

indicator variable

Write $B(n,p) = X_1 + X_2 + \cdots + X_n$

$$X_i = \begin{cases} 1 & i\text{th coin is } H \\ 0 & \text{otherwise} \end{cases}$$

$$E[B(n,p)] = E(X_1) + E(X_2) + \cdots + E(X_n)$$
$$= nE(X_1) = n(1 \times p + 0 \times (1-p)) = np$$

Same probability Space $\{H,T\}^n$

Suppose Z = # of times we get HH

$$Z(HTHHTTHHHT) = 3$$

$$Z_i = \begin{cases} 1 & \text{if } i, i+1 \text{ coin } H \\ 0 & \text{otherwise} \end{cases}$$

$$Z = Z_1 + Z_2 + \cdots + Z_{n-1}$$

$$E(Z) = E(Z_1) + \cdots + E(Z_{n-1})$$
$$= p^2 + \cdots + p^2$$
$$= (n-1)p^2.$$

m indistinguishable balls

n colors

Z = # colors used.

$$Z_i = \begin{cases} 1 & \text{color } i \text{ is used} \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} E(Z) &= n\,E(Z_1) \\ &= n\,\Pr(Z_1 \neq 0) \\ &= n\left(1 - \Pr(Z_1 = 0)\right) \\ &= n\left(1 - \frac{\binom{n+m-2}{m}}{\binom{n+m-1}{m}}\right) \end{aligned}$$

colorings not using color 1

colorings

$$= n\left(1 - \frac{\binom{n+m-2}{m}}{\binom{n+m-1}{m}}\right)$$

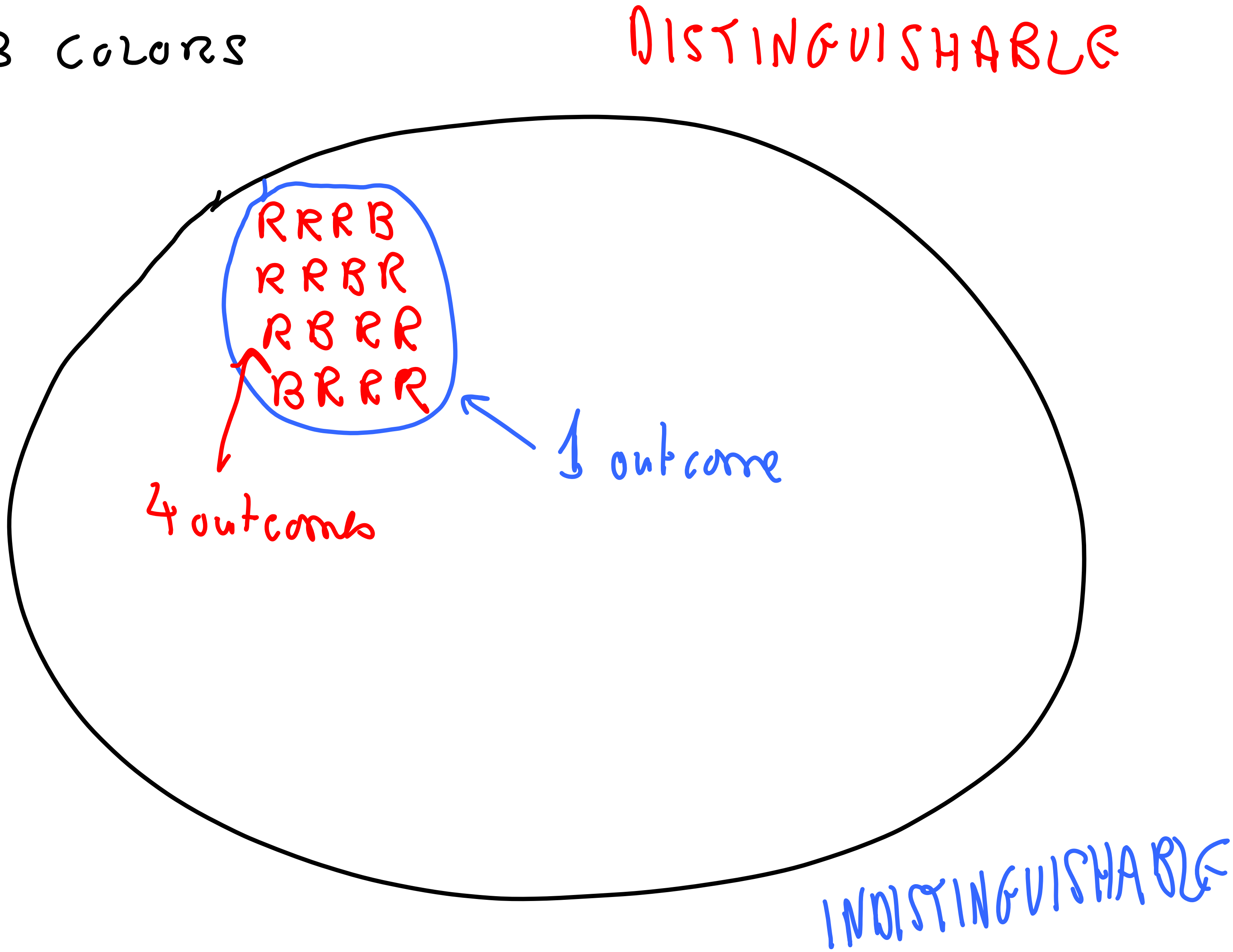
$$= n\left(1 - \frac{(n+m-2)!}{m!\,(n-2)!} \times \frac{m!\,(n-1)!}{(n+m-1)!}\right)$$

$$= n\left(1 - \frac{n-1}{n+m-1}\right)$$

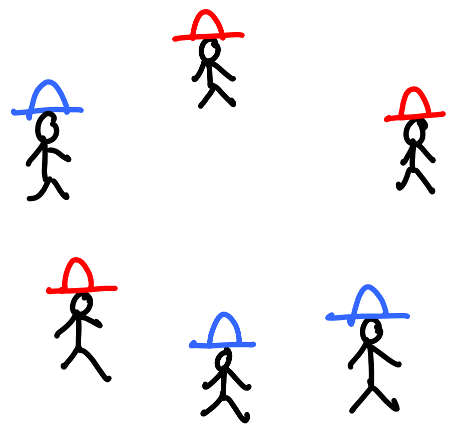
$$= \frac{mn}{n+m-1}$$

4 BALLS

3 COLORS



HAT PUZZLE



(i) i can see the hat of j, $\forall j \neq i$.

(ii) The color of the hats are random.

Each person says (i) My hat is RED

or (ii) My hat is BLUE

or (iii) I don't know.

Big prize if one person correctly guesses own hat color & no-one is wrong.

Big penalty if someone is wrong.

Is there a strategy that wins with "high" prob.

Claim: Can win with probability $\geq 1 - \frac{2\log n}{n}$.

Represent hat colors by $\{0,1\}^n \leftarrow Q_n$

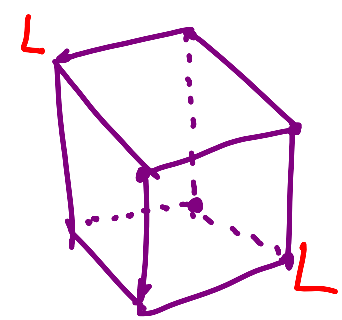
$$Q_n = \underset{\text{win}}{W} \cup \underset{\text{loss}}{L}$$

L has property that if $x = x_1 x_2 \dots x_n \notin L$ then can change one bit of x, i say, so that

$x_1 x_2 \dots x_{i-1} \, x_i' \, x_{i+1} \dots x_n \in L$.

$\uparrow$

$1 - x_i$



Before the hats are placed; People agree on an L. (remember).

Person i: $x_i = ?$

x_1

x_2

$\vdots$

x_{i-1}

x_{i+1}

$\vdots$

x_n

i will assume $x \in W$.

If, say, $x_1 x_2 \cdots x_{i-1} 0 x_{i+1} \cdots x_n \in W$

$x_1 x_2 \cdots x_{i-1} 1 x_{i+1} \cdots x_n \in L$

i will say $x_i = 0$.

How to choose L.

$$p = \frac{\log n}{n}$$

Choose L_1 randomly by placing $x \in Q_n$ into L_1 with probability p.

L_2 = those x not covered by L_1

$L = L_1 \cup L_2$

$$E(|L|) = \underbrace{2^n p}_{E(|L_1|)} + \underbrace{2^n (1-p)^{n+1}}_{E(|L_2|)}$$