10\3\2007

Boole's Inequality

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\leq P(A) + P(B)$$

If $A \cap B = \emptyset \Rightarrow$ disjoint events.

$$P(A \cup B) = P(A) + P(B)$$

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) \leq P(A_1) + P(A_2) + \cdots + P(A_n)$$

Simple induction:

Set Coloring Problem

$A_1, A_2, \ldots, A_n$ are all subsets of A.

$|A_i| = k, \quad i = 1, 2, \ldots, n.$

A 2-coloring is a partition of $A = \underset{\text{red}}{R} \cup \underset{\text{blue}}{B}$

s.t.

$$A_i \cap R \neq \emptyset \quad i = 1, 2, \ldots n$$

$$A_i \cap B \neq \emptyset \quad i = 1, 2, \ldots n$$

$A_1 = \{3, 4, 5\}$

$A_2 = \{1, 2, 4\}$

$A_3 = \{2, 4, 5\}$

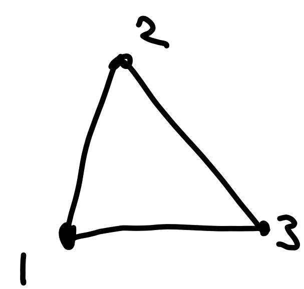
$R = \{1, 2, 3\}$

$B = \{4, 5\}$

$A_1 = \{1, 2\}$

$A_2 = \{2, 3\}$ — cannot 2-color.

$A_3 = \{1, 3\}$

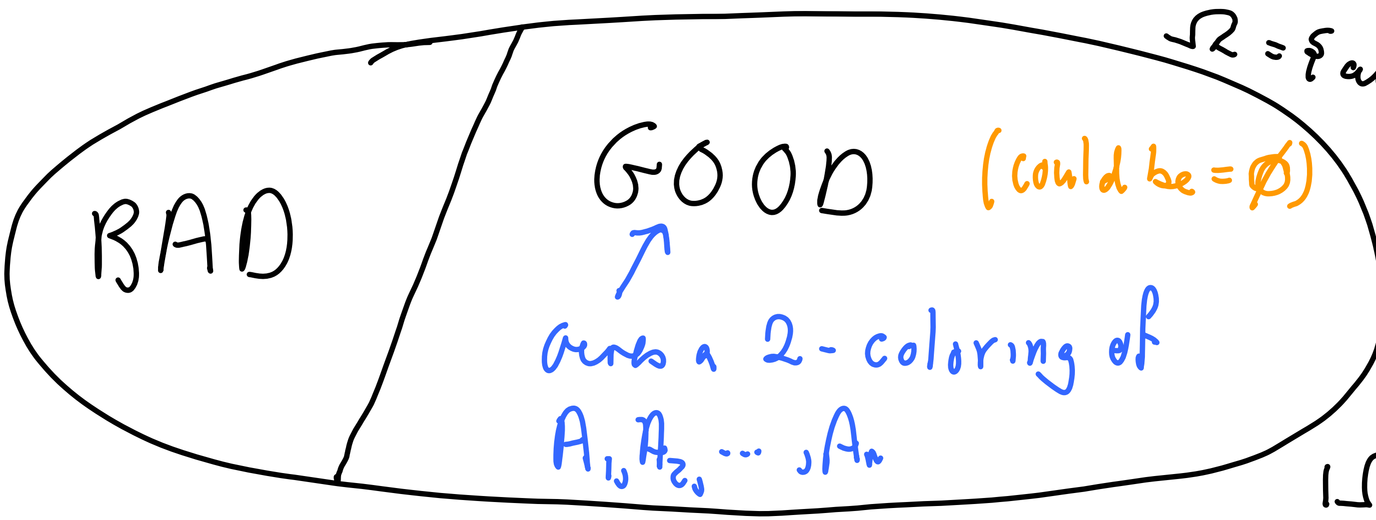


Theorem

$|A_i| = k$, $i = 1, 2, \dots n$ and $n < 2^{k-1}$

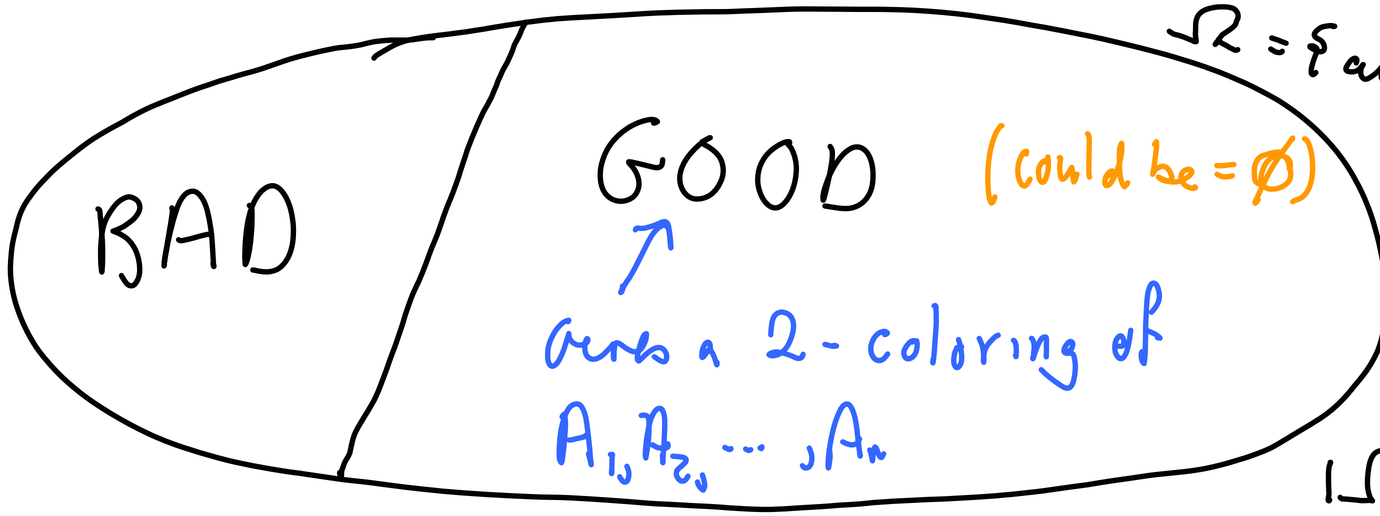
$\Rightarrow \exists$ a 2-coloring.

Probabilistic Method: Erdős



$\Omega = \{\text{all colorings}\}$

$|\Omega| = 2^{|A|}$



$\Omega = \{\text{all colorings}\}$

$|\Omega| = 2^{|A|}$

Theorem states that GOOD $\neq \emptyset$

Turn Ω into a probability space with uniform distribution — each coloring equally likely. Need to show $P(\text{GOOD}) > 0$

or $P(\text{BAD}) < 1$

$$BAD = \bigcup_{i=1}^{n} \underbrace{BAD(i)}_{A_i \text{ is mono-colored.}}$$

$$P(BAD) \leq \sum_{i=1}^{n} P(BAD(i))$$

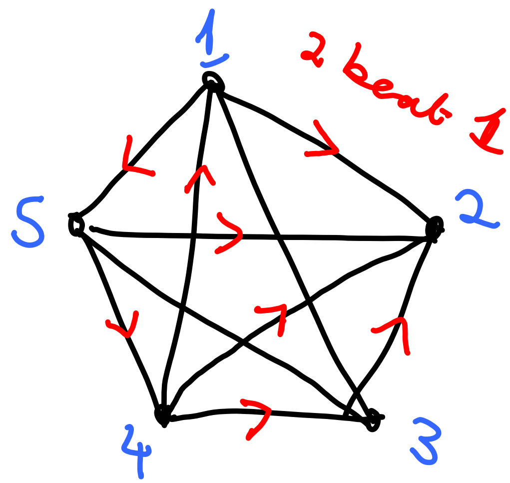
$$= \sum_{i=1}^{n} \frac{1}{2^{k-1}}$$

$$= \frac{n}{2^{k-1}}$$

$$< 1.$$

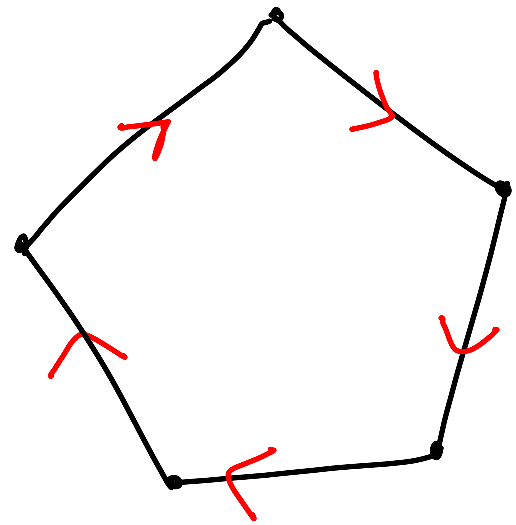
Tournaments

n players. Each play each other.



Fix k. Is it possible for <u>every</u> set S of k players there exists person w_S who beats all players in S.

$k = 1$



For arbitrary k we use the probabilistic method. Let n be large and let T be a random tournament. Arrows are randomly chosen.

For $|S| = k$, let $\mathcal{E}_S$ be "nobody beats everyone in S"

$$\mathcal{E} = \bigcup_S \mathcal{E}_S$$

we must show that there exist tournaments in which none of the $\mathcal{E}_S$ occur.

$$\Pr(\mathcal{E}) = \Pr\Big(\bigcup_{S} \mathcal{E}_S\Big)$$

$$\leq \sum_{S} \Pr(\mathcal{E}_S)$$

$$= \sum_{S} \left(1 - \frac{1}{2^k}\right)^{n-k}$$

S $\quad\quad$ $\cdot\, v$

$\Pr(v \text{ beats } S) = \frac{1}{2^k}$

$\Pr(v \text{ does not beat } S) = 1 - \frac{1}{2^k}$

$\Pr(\text{No } v \text{ beats } S) = \left(1 - \frac{1}{2^k}\right)^{n-k}$

$$\Pr(\mathcal{E}) \leq \sum_{S} \left(1 - \frac{1}{2^k}\right)^{n-k}$$

$$= \binom{n}{k}\left(1 - \frac{1}{2^k}\right)^{n-k}$$

k is fixed
n is large

$$\leq n^k e^{-(n-k)/2^k}$$

$$= \exp\left\{k \log n - \frac{n-k}{2^k}\right\}$$

$$\longrightarrow 0$$