10/01/2007

Probability Space:

Set Ω

Ω infinite or countable.

$P : \Omega \to \mathbb{R}^+$

$$\sum_{\omega \in \Omega} P(\omega) = 1$$

Probability of ω

Geometric Distribution

$$P(k) = (1-p)^{k-1}p$$

Event:

$A \subseteq \Omega$ is a "event".

$$P(A) = \sum_{\omega \in A} P(\omega).$$

Lottery

$$|\Omega| = \binom{80}{11}$$

$$|WIN| = \binom{73}{4}$$

Birthday Paradox

d people

$\Omega = [n]^d$ = sequence of birthdays

$D = \{$ distinct birthday $\}$

$1 + x \leq e^{x}$

$$P[D] = \frac{n \times (n-1) \times (n-2) \times \cdots \times (n-d+1)}{\underbrace{n \times n \times n \times \cdots \times n}_{d \text{ times}}}$$

$$= \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{d-1}{n}\right) \leq e^{-\left(\frac{1}{n} + \frac{2}{n} + \cdots + \frac{d-1}{n}\right)}$$

$$= e^{-\frac{d(d-1)}{2n}}$$

Balls in Boxes

Placing m distinguishable balls randomly into n boxes.

$\Omega = [n]^m = b_1, b_2, b_3, \dots, b_m$

Ball i is placed in box b_i

$E = \{\text{box } 1 \text{ is empty}\}$

$P(E) = (1 - \frac{1}{n})^m \longrightarrow e^{-c}$ if $m = cn$

$$1 + x \le e^{x}$$

(i) $x \ge 0$: $\quad e^{x} = 1 + x + \frac{x^2}{2} + \cdots \ge 1 + x$

(ii) $x \le -1$: $\quad 1 + x \le 0 \le e^{x}$

(iii) $-1 < x < 0$ $\quad 1 - y \le 1 - y + \underbrace{\frac{y^2}{2} - \frac{y^3}{6}}_{\frac{y^2}{2}\left(1 - \frac{y}{3}\right)} + \cdots$

$x = -y$

$$\left(1 - \frac{1}{n}\right)^{cn} \le \left(e^{-1/n}\right)^{cn} = e^{-c}$$

$$e^{-x-x^2} \leq 1-x \quad , \quad x \leq \frac{1}{100}$$

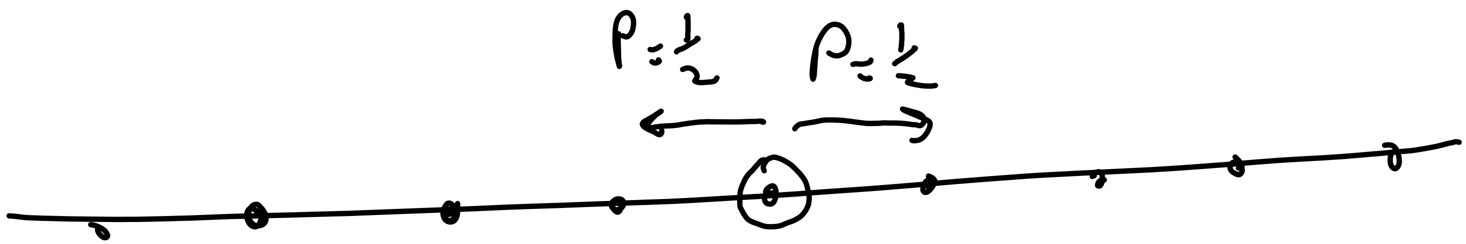
$$\log_e(1-x) = -x - \frac{x^2}{2} - \cdots$$

$$\geq -x - x^2$$

$$\left(1-\frac{1}{n}\right)^{cn} \geq e^{-\left[\frac{1}{n}+\frac{1}{n^2}\right]cn}$$

$$\longrightarrow e^{-c}$$

Random Walk



$\Omega = \{L, R\}^n$ with uniform distribution

n steps.

X_i = position after i steps

$$\Pr[X_n = 0] = \begin{cases} 0 & n \text{ is odd} \\ \dfrac{\binom{n}{m}}{2^n} & n = 2m \end{cases}$$

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$\omega \in \Omega$

$$\underbrace{LRRLLRRLLRRLR\cdots}_{n}$$

Back at 0 : $\#R = \#L$

Stirling's Formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

$$\binom{n}{m} = \frac{n!}{\frac{n}{2}!\cdot\frac{n}{2}!}$$