21-301 Combinatorics
Homework 9
Due: Wednesday, November 29

1. Prove that if we 2-color the edges of K_6 then there are least *two* mono-chromatic triangles.

   Solution Assume w.l.o.g. that triangle $(1, 2, 3)$ is Red and that $(4, 5, 6)$ is not Red and in particular that edge $(4, 5)$ is Blue. If $x = 4, 5$ or 6 then there can be at most one Red edge joining x to $1, 2, 3$, else we get a Red triangle. So we can assume that there are two Blue edges joining each of $4, 5$ to $1, 2, 3$. So there must be $x \in \{1, 2, 3\}$ such that both $(x, 4)$ and $(x, 5)$ are Blue. But then triangle $(x, 4, 5)$ is Blue.

2. Prove that if $n \geq R(2k, 2k)$ and if we 2-color the edges of $K_{n,n}$ then there is a mono-chromatic copy of $K_{k,k}$.

   Solution Given a coloring σ of $K_{n,n}$ we construct a coloring τ of the edges of K_n as follows. If $i < j$ then we give the edge (i, j) of K_n the same color that is given to edge (i, j) under σ.

   Since $n \geq R(2k, 2k)$ we see that K_n contains a mono-colored copy of K_{2k}. If the set of vertices of this copy is S, divide S into two parts S_1, S_2 of size k where $\max S_1 < \min S_2$. It follows that the bipartite sub-graph of $K_{n,n}$ defined by S_1, S_2 is mono-colored under σ.

3. Let $I_1, I_2, \ldots, I_{mn+1}$ be closed intervals on the real line i.e. $I_j = [a_j, b_j]$ where $a_j \leq b_j$ for $1 \leq j \leq mn + 1$. Use Dilworth's theorem to show that either (i) there are $m + 1$ intervals that are pair-wise disjoint or (ii) there are $n + 1$ intervals with a non-empty intersection.

   Solution Define a partial ordering $<$ on the intervals by $I_r < I_s$ iff $b_r < a_s$. Suppose that $I_{i_1}, I_{i_2}, \ldots, I_{i_t}$ is a collection of pair-wise disjoint intervals. Assume that $a_{i_1} < a_{i_2} \cdots < a_{i_t}$. Then $I_{i_1} < I_{i_2} \cdots < I_{i_t}$ form a chain and conversely a chain of length t comes from a set of t pair-wise disjoint intervals. So if (i) does not hold, then the maximum length of a chain is m. This means that the minimum number of chains needed to cover the poset is at least $\left\lceil \frac{mn+1}{m} \right\rceil = n+1$. Dilworth's theorem implies that there must exist an anti-chain $\{I_{j_1}, I_{j_2}, \ldots, I_{j_{n+1}}\}$. Let $a =$

$\max\{a_{j_1}, a_{j_2}, \ldots, a_{j_{n+1}}\}$ and $b = \min\{b_{j_1}, b_{j_2}, \ldots, b_{j_{n+1}}\}$. We must have $a \leq b$ else the intervals giving a, b are disjoint. But then every interval of the anti-chain contains $[a, b]$.