21-301 Combinatorics
Homework 8
Due: Wednesday, November 8

1. Let $s \geq 1$ be fixed. Let $\mathcal{A}$ be a family of subsets of $[n]$ such that **there do not exist** distinct $A_1, A_2, \ldots, A_{s+1}$ such that $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_{s+1}$. Show that
$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq s.$$
Solution Let π be a permutation of $[n]$ and for $A \in \mathcal{A}$ let
$$1_A = \begin{cases} 1 & if\ \{\pi(1), \pi(2), \ldots, \pi(|A|)\} = A \\ 0 & otherwise \end{cases}$$
Our condition ensures that for all π,
$$\sum_{A \in \mathcal{A}} 1_A \leq s.$$
Now let π be a random permutation. Then
$$s \geq \mathbf{E}\left(\sum_{A \in \mathcal{A}} 1_A\right) = \sum_{A \in \mathcal{A}} \mathbf{E}(1_A) = \sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}}.$$

2. Suppose that $2^m > mn - \binom{m}{2} + 1$ and S, $|S| = m$ is a subset of $[n]$. Show that there exist distinct $A, B \subseteq S$ such that (i) $A \cap B = \emptyset$ and (ii) $\sum_{a \in A} a = \sum_{b \in B} b$.

   Solution Fix S and for $T \subseteq S$ let $a_T = \sum_{a \in T} a$. The integers a_T satisfy $0 \leq a_T \leq mn - \binom{m}{2}$ and since there are 2^m of them, by the pigeon-hole principle, there exist distinct T_1, T_2 such that $a_{T_1} = a_{T_2}$. Now let $X = T_1 \cap T_2$ and $A = T_1 \setminus X$, $B = T_2 \setminus X$. Then A, B satisfy the requested conditions.

3. Given any sequence of n integers, positive or negative, not necessarily all different, show that some consecutive subsequence has the property that the sum of the members of the subsequence is a multiple of n.

Solution Let the sequence be $x_1, x_2, \ldots, x_n$ and let $s_i = x_1 + \cdots + x_i \mod n$ for $i = 1, 2, \ldots, n$. If there exists i with $s_i = 0$ then n divides $x_1 + \cdots + x_i$. Otherwise, $s_1, s_2, \ldots, s_n$ all take values in $[n-1]$. By the pigeon-hole principle, there exist $i < j$ such that $s_i = s_j$ and then n divides $x_{i+1} + \cdots + x_j$.