21-301 Combinatorics
Homework 7
Due: Wednesday, November 1

1. A tree T has exactly one vertex of degree i for each $2 \leq i \leq m$ and all other vertices are of degree one. How many vertices does T have? Justify your answer.

   Solution Let n be the number of vertices in T and k be the number of vertices of degree 1. Then

   $$\begin{aligned} n &= k + m - 1 \\ 2n - 2 &= k + 2 + \cdots + m = k + \frac{m(m+1)}{2} - 1 \end{aligned}$$

   The second equation says that 2 times number of edges = sum of degrees in T.

   Solving, we obtain
   $$n = \frac{m^2 - m + 4}{2}.$$

2. Let $A_1, A_2, \ldots, A_n$ be n distinct subsets of $[n]$. Show that there is an element $x \in [n]$ such that all of the sets $A_i \setminus \{x\}$ are also distinct.

   Hint: Consider the graph G with *vertices* $A_1, A_2, \ldots, A_n$ and an edge $\{A_i, A_j\}$ with "colour" x whenever $A_i \oplus A_j = \{x\}$. Prove that in any cycle of G, each colour appears an even number of times. Deduce that one can delete edges of G so that no cycles are left and the number of colours remains the same.

   Solution We want to show that at most $n - 1$ colours appear on the edges of G. If we know that x does not appear as a colour, then all of the sets $A_i \setminus \{x\}$ are also distinct. ($A_i \setminus \{x\} = A_j \setminus \{x\}$ iff $A_i = A_j$ (not allowed) or $A_i \oplus A_j = \{x\}$).

   Let $C = A_{i_1}, A_{i_2}, \ldots, A_{i_k}, A_{i_1}$ be a cycle in G and suppose that $A_{i_t} \oplus A_{i_{t+1}} = \{x_t\}$ for $t = 1, 2, \ldots k$. Then since $X \oplus X = \emptyset$ for any X, we have

$$\bigoplus_{t=1}^{k}(A_{i_t} \oplus A_{i_{t+1}}) = \emptyset.$$

This implies that

$$\bigoplus_{t=1}^{k}\{x_t\} = \emptyset.$$

So that each "colour" appears an even number of times on the edges of a cycle.

So, removing one edge from a cycle does not reduce the number of edge colours in the graph. Repeating this until there are no cycles, we see that we have the same number of edge colours and a graph with at most $n-1$ edges.

3. Show that a sequence $(d_1, d_2, \ldots, d_n)$ of positive integers is the degree sequence of a tree if and only if $\sum_{i=1}^{n} d_i = 2(n-1)$.

   Solution If a sequence $(d_1, \ldots, d_n)$ is a degree sequence of a tree $T = (V, E)$, then $\sum_{i=1}^{n} d_i = 2|E|$, but in a tree, $|E| = n-1$, so $\sum_{i=1}^{n} d_i = 2(n-1)$ (this is the "only if" part of the claim).

   To show the "if" direction, we must construct a tree having the given degree sequence. This is trivial if $n = 2$, since then $d_1 = d_2 = 1$ and we can take a single edge as our tree with this degree sequence. There must be an i such that $d_i = 1$, else $\sum_{i=1}^{n} d_i \geq 2n$. So assume that $d_n = 2$. There must be an i such that $d_i \geq 2$, else $\sum_{i=1}^{n} d_i = n < 2(n-1)$. Assume that $d_{n-1} \geq 2$. Now consider the sequence $(d'_1 = d_1, \ldots, d'_{n-2} = d_{n-2}, d'_{n-1} = d_{n-1} - 1)$. We have $\sum_{i=1}^{n-1} d'_i = 2(n-2)$ and so, by induction on n, there is a tree T' with the degree sequence $(d'_1, \ldots, d'_{n-1})$. But then adding vertex n and the edge $(n-1, n)$ to T' creates a tree T with degree sequence $(d_1, d_2, \ldots, d_n)$.