21-301 Combinatorics
Homework 7
Due: Wednesday, November 1

1. A tree T has exactly one vertex of degree i for each $2 \leq i \leq m$ and all other vertices are of degree one. How many vertices does T have? Justify your answer.

2. Let $A_1, A_2, \ldots, A_n$ be n distinct subsets of $[n]$. Show that there is an element $x \in [n]$ such that all of the sets $A_i \setminus \{x\}$ are also distinct.

   Hint: Consider the graph G with *vertices* $A_1, A_2, \ldots, A_n$ and an edge $\{A_i, A_j\}$ with "colour" x whenever $A_i \oplus A_j = \{x\}$. Prove that in any cycle of G, each colour appears an even number of times. Deduce that one can delete edges of G so that no cycles are left and the number of colours remains the same.

3. Show that a sequence $(d_1, d_2, \ldots, d_n)$ of positive integers is the degree sequence of a tree if and only if $\sum_{i=1}^{n} d_i = 2(n-1)$.