21-301 Combinatorics
Homework 6
Due: Wednesday, October 25

1. Let G be a simple graph with $n > 3$ vertices and no vertex of degree $n-1$. Suppose that for any two vertices of G there is a **unique** vertex joined to both of them. Prove

   (i) If x and y are non-adjacent then they have the same degree.

   (ii) Now show that G is a **regular** graph i.e. every vertex has the same degree.

   [Hint for (i): Suppose that the x, y are non-adjacent. Let the neigbours of x be $x_1, x_2, \ldots, x_k$ and the neighbours of y be $y_1, y_2, \ldots, y_l$, with $x_1 = y_1$. Show that each y_i is adjacent to a unique member of $\{x_1, x_2, \ldots, x_k\}$.]

2. A sequence $\mathbf{d} = d_1 \geq d_2 \geq \cdots \geq d_n$ is said to be *graphic* if there is a simple graph with degree sequence $\mathbf{d}$. Show that:

   (a) The sequences (7,6,5,4,3,3,2) and (6,6,5,4,3,3,1) are not graphic.

   (b) If $\mathbf{d}$ is graphic then

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\} \qquad for\ 1 \leq k \leq n.$$

3. Show that a simple graph $G = (V, E)$ has a bipartite spanning subgraph H such that $d_H(v) \geq d_G(v)/2$ for all $v \in V$.
   [Hint: Consider a bipartite spanning subgraph with as many edges as possible.]