21-301 Combinatorics
Homework 4
Due: Wednesday, October 4

1. n distinguishable balls are independently and randomly numbered with 1,2,3 or 4, each number being equally likely. What is the probability that the number of balls with color 1 is equal to the number of balls with color 2. (The answer is a sum).

   Solution Suppose the common number is k. There are $\binom{n}{k,k,n-2k}$ ways of choosing two sets A, B of size k and giving them colors 1 and 2 respectively. The probability that A is colored with 1 and B is colored with 2 and $[n] \setminus (A \cup B)$ is colored with 3 and 4 is $\frac{1}{4^{2k}2^{n-2k}}$. Thus the answer is

   $$\sum_{k=0}^{n} \binom{n}{k, k, n-2k} \frac{1}{4^{2k}2^{n-2k}}.$$

2. Let $A_1, A_2, \ldots, A_m$ be subsets of A and $|A_i| = n$. Show that if $m < 3^{n-1}$ then there is a way of coloring A with 3 colors so that each set gets at least two colours.

   Solution Let $\mathcal{E}_i$ be the event that A_i is mono-colored. Then

   $$\Pr(\mathcal{E}_i) = 3^{1-n}.$$

   Thus,

   $$\Pr\left(\bigcup_{i=1}^{m} \Pr(\mathcal{E}_i)\right) \leq m3^{1-n} < 1$$

   and so there is a coloring for which none of the $\mathcal{E}_i$ occur.

3. Let $s_1, s_2, \ldots, s_m$ be ternary strings such that no string is a prefix of another string.
   ($a = a_1a_2 \cdots a_p$ is a prefix of $b = b_1b_2 \cdots b_q$ if $p \leq q$ and $a_i = b_i$ for $1 \leq i \leq p$).

   Show that

   $$\sum_{i=1}^{m} 3^{-|s_i|} \leq 1$$

where $|s|$ is the length of string s.

(Hint: Let $n = \max\{|s_1| : 1 \leq i \leq n\}$. Let x be a random ternary string of length n. Consider the events $\mathcal{E}_i = \{s_i$ is a prefix of x.)

Solution The events $\mathcal{E}_i$ of the hint are *disjoint*. This follows from the assumption that no string is a prefix of another. Thus

$$1 \geq \sum_{i=1}^{m} \Pr(\mathcal{E}_i) = \sum_{i=1}^{m} 3^{-|s_i|}.$$