21-301 Combinatorics
Homework 4
Due: Wednesday, October 4

1. n distinguishable balls are independently and randomly numbered with 1,2,3 or 4, each number being equally likely. What is the probability that the number of balls with color 1 is equal to the number of balls with color 2. (The answer is a sum).

2. Let $A_1, A_2, \ldots, A_m$ be subsets of A and $|A_i| = n$. Show that if $m < 3^{n-1}$ then there is a way of coloring A with 3 colors so that each set gets at least two colours.

3. Let $s_1, s_2, \ldots, s_m$ be ternary strings such that no string is a prefix of another string.
   ($a = a_1 a_2 \cdots a_p$ is a prefix of $b = b_1 b_2 \cdots b_q$ if $p \leq q$ and $a_i = b_i$ for $1 \leq i \leq p$).
   Show that
   $$\sum_{i=1}^{m} 3^{-|s_i|} \leq 1$$
   where $|s|$ is the length of string s.
   (Hint: Let $n = \max\{|s_i| : 1 \leq i \leq m\}$. Let x be a random ternary string of length n. Consider the events $\mathcal{E}_i = \{s_i \text{ is a prefix of } x\}$.)