

21-301 Combinatorics  
Homework 3  
Due: Monday, September 19

1. A group of  $n$  people is seated around a round table. The group leaves the table for a break and then returns. In how many ways can the people sit down after returning so that no one has the same person sitting to their left as before. Formally, if  $\pi(i)$  is the person now in position  $i$ , then  $\pi(i+1) \neq \pi(i) + 1$  for  $i = 1, 2, \dots, n$ . (Interpret  $n+1$  as 1).

**Solution:** Label the table positions as  $1, 2, \dots, n$  as well as the people. Let  $A_i$ ,  $1 \leq i \leq n$  be the set of placings of the people so that  $\pi(i+1) = \pi(i) + 1$ . We need to determine  $|A_S|$  for  $S \subseteq [n]$ .

We first observe that  $A_{[n]} = A_{[n-1]}$ . For if  $\pi(i+1) = \pi(i)$  for  $1 \leq i \leq n-1$  then  $\pi(n) = \pi(1) + n - 1 \pmod n = \pi(1) - 1 \pmod n$ .

Suppose next that  $|S| = k \leq n-1$ , then

$$|A_S| = n(n-k-1)!. \quad (1)$$

A *segment* of  $S$  is a maximal subset that consists of consecutively numbered elements e.g.  $S = \{2, 3, 4, 7, 8, 11, 12, 14\}$  consists of segments  $\{2, 3, 4\}$ ,  $\{7, 8\}$ ,  $\{11, 12\}$  and  $\{14\}$ . Suppose there are  $p$  segments of sizes  $k_1, k_2, \dots, k_p$  where  $k_1 + \dots + k_p = k$ . To obtain a permutation in  $A_S$  we take each segment  $[i, j]$  and shrink the set  $[i, j+1]$  into a single object (notice  $j+1$  in place of  $j$ ). This gives  $p$  objects and we get a further  $n-k-p$  objects from the elements of  $[n]$  which have are not one more than the largest element of a segment. There are now  $n-k$  objects to permute. We choose the place to put the first interval in  $n$  ways and then there are  $(n-k-1)!$  ways to place the remaining objects. This proves (1).

Thus, the number of placings is

$$\sum_{k=0}^{n-1} \sum_{|S|=k} (-1)^k n(n-|S|-1)! = n \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} (n-k-1)!.$$

2. Show that the number of permutations of  $[n]$  which do not contain a consecutive pair of the form  $k, k+1$  satisfies the recurrence

$$b_n = (n-1)b_{n-1} + (n-2)b_{n-2}.$$

[Hint: delete  $n$  from such a sequence and separately count those permutations which still satisfy the condition and those that don't.]

**Solution:** Let

$$B_n = \{\pi : \nexists \text{ consecutive pair } k, k+1, 1 \leq k < n\}$$

so that  $b_n = |B_n|$ .

If we remove  $n$  from the sequence  $\pi(1), \pi(2), \dots, \pi(n)$  then we obtain a permutation  $\hat{\pi}$  of  $[n-1]$ . Let

$$B'_n = \{\pi \in B_n : \hat{\pi} \in B_{n-1}\} \text{ and } B''_n = B_n \setminus B'_n.$$

Let

$$B'_{n,k} = \{\pi \in B'_n : n \text{ follows } k\}$$

where  $k = 0, 1, \dots, n-2$ . ( $n$  follows 0 means that it goes at the front).

The map “place  $n$  after  $k$ ” defines a bijection between  $B_{n-1}$  and  $B'_{n,k}$  and so  $|B'_{n,k}| = |B_{n-1}| = b_{n-1}$ . (The inverse of this map is “remove  $n$ ”).

Next let

$$B''_{n,k} = \{\pi \in B''_n : n \text{ lies between } k \text{ and } k+1\}$$

where  $k = 1, \dots, n-2$ .

If, in the sequence defined by  $\pi$ , we remove  $n$  and replace  $k, k+1$  by  $k$  and  $i > k$  by  $i-1$  then we obtain a member of  $B_{n-2}$ . The inverse of this map is to replace  $i > k$  by  $i+1$  and then replace  $k$  by the sequence  $k, n, k+1$ . Thus,  $|B''_{n,k}| = |B_{n-2}| = b_{n-2}$  and we get our recurrence.

3. Let  $a_0, a_1, a_2, \dots$  be the sequence defined by the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2} + 1 \quad \text{for } n \geq 2$$

with initial conditions  $a_0 = 1$  and  $a_1 = 3$ . Determine the generating function for this sequence, and use the generating function to determine  $a_n$  for all  $n$ .

**Solution:**

$$\begin{aligned} \sum_{n=2}^{\infty} (a_n - a_{n-1} - 2a_{n-2})x^n &= - \sum_{n=2}^{\infty} x^n \\ a(x) - 1 - 3x - x(a(x) - 1) - 2x^2a(x) &= - \frac{x^2}{1-x} \\ a(x)(1-x-2x^2) &= 1+2x - \frac{x^2}{1-x} \end{aligned}$$

$$\begin{aligned} a(x) &= \frac{1+2x}{(1+x)(1-2x)} - \frac{x^2}{(1-x^2)(1-2x)} \\ &= \frac{1+x-x^2}{(1-x)(1+x)(1-2x)} \\ &= -\frac{1/6}{1+x} - \frac{1/2}{1-x} + \frac{5/3}{1-2x} \\ &= \sum_{n=0}^{\infty} \left( -\frac{1}{6}(-1)^n - \frac{1}{2} + \frac{5}{3}2^n \right) x^n. \end{aligned}$$

So

$$a_n = -\frac{1}{6}(-1)^n - \frac{1}{2} + \frac{5}{3}2^n \quad \text{for } n \geq 0.$$