21-301 Combinatorics
Homework 1
Due: Wednesday, September 13

1. Show that
$$\sum_{k=1}^{n} \binom{n}{k} \frac{k}{n^k} = \left(1 + \frac{1}{n}\right)^{n-1}.$$
[Hint: Differentiate the expression in the binomial theorem and then put in a suitable value for x.]
Solution: Differentiating,
$$n(1+x)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k x^{k-1}.$$
Putting $x = 1/n$ we get
$$n\left(1 + \frac{1}{n}\right)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} \frac{k}{n^{k-1}}.$$
Dividing through by n gives the answer.

2. Show that for a fixed k,
$$\sum_{\ell=0}^{n-k} \binom{n}{k, \ell, n-k-\ell} 2^{\ell} = 3^{n-k} \binom{n}{k}. \tag{1}$$
[Hint: Expand $(1+x+y)^n$ using the multinomial theorem. Then put $y = 1$ and extract the coefficient of x^k in what remains.]
Solution: From the multinomial theorem
$$(1+x+y)^n = \sum_{k=0}^{n} \left(\sum_{\ell=0}^{n-k} \binom{n}{k, \ell, n-k-\ell} y^{\ell} \right) x^k.$$
Putting $y = 2$ we get
$$(3+x)^n = \sum_{k=0}^{n} \left(\sum_{\ell=0}^{n-k} \binom{n}{k, \ell, n-k-\ell} 2^{\ell} \right) x^k.$$
But
$$(3+x)^n = \sum_{k=0}^{n} 3^{n-k} \binom{n}{k} x^k$$
and so we obtain (1).

3. Find an expression for the size of the set

$$\{(x_1, x_2 \ldots, x_m)\} \in Z^m : \ x_1 + x_2 + \cdots + x_m = n \ and \ 0 \le x_j \le a \ for \ j = 1, 2, \ldots, m\}.$$

[You should use Inclusion-Exclusion and expect to have your answer as a sum.]

Solution: Let

$$A = \{(x_1, x_2 \ldots, x_m) \in Z^m : \ x_1 + x_2 + \cdots + x_m = n \ and \ 0 \le x_j \ for \ j = 1, 2, \ldots, m\}.$$

Let

$$A_i = \{(x_1, x_2 \ldots, x_m) \in A : \ x_i \ge a + 1\}$$

for $i = 1, 2, \ldots, m$.

We are asked for the size of

$$\bigcap_{i=1}^{m} \bar{A}_i = \sum_{S \subseteq [m]} (-1)^{|S|} |A_S|.$$

Now for $S \subseteq [m]$,

$$|A_S| =$$
$$|\{(x_1, x_2 \ldots, x_m) \in A : \ x_i \ge a + 1, \ i \in S\}| =$$
$$|\{(y_1, y_2 \ldots, y_m) \in Z^m : \ y_1 + x_2 + \cdots + y_m = n - (a + 1)|S| \ and \ 0 \le y_j \ for \ j = 1, 2, \ldots, m\} =$$
$$\binom{m + n - (a + 1)|S| - 1}{m - 1}.$$

So, the size of the set in question is

$$\sum_{|S| \subseteq [m]} (-1)^{|S|} \binom{m + n - (a + 1)|S| - 1}{m - 1} = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \binom{m + n - (a + 1)k - 1}{m - 1}.$$