21-301 Combinatorics
Homework 1: Solutions
Due: Wednesday, September 7

1. How many integral solutions of

$$x_1 + x_2 + x_3 + x_4 = 40$$

satisfy $x_1 \geq 2$, $x_2 \geq 0$, $x_3 \geq -5$ and $x_4 \geq 5$?

Solution Let

$$y_1 = x_1 - 2, \qquad y_2 = x_2, \qquad y_3 = x_3 + 5, \qquad y_4 = x_4 - 5.$$

An integral solution of $x_1 + x_2 + x_3 + x_4 = 40$ such that $x_1 \geq 2$, $x_2 \geq 0$, $x_3 \geq -5$ and $x_4 \geq 5$ corresponds to an integral solution of $y_1 + y_2 + y_3 + y_4 = 38$ such that $y_1, \dots, y_4 \geq 0$. From a result in class,

$$|\{(y_1, y_2, y_3, y_4) : y_1, \dots, y_4 \in Z_+ \text{ and } y_1 + \dots + y_4 = 38\}| = \binom{38 + 4 - 1}{4 - 1} = \binom{41}{3}.$$

2. Prove the following equality using a *combinatorial* argument

$$\sum_{i=1}^{n} \binom{i}{2}\binom{n}{i} = \binom{n}{2} 2^{n-2}.$$

Solution Consider the set

$$\mathcal{S} = \{(A, \{x, y\}) : A \subseteq [n] \quad \text{and} \quad \{x, y\} \subseteq A\}.$$

In words, $\mathcal{S}$ is the set of all ordered pairs consisting of a subset of $[n]$ and two elements of that set. We count the elements of $\mathcal{S}$ is two ways.

First we count with respect to the elements x, y. There are $\binom{n}{2}$ choices for x, y. Once x, y are fixed, $A \setminus \{x, y$ can be any subset of $[n] \setminus \{\{x, y\}\}$. There are 2^{n-2} such sets. Therefore, we have

$$|\mathcal{S}| = n2^{n-2}.$$

Now we count with respect to the set A. For $i = 1, 2, \dots, n$ let

$$\mathcal{S}_i = \{(A, \{x, y\}) \in \mathcal{S} : |A| = i\}.$$

These sets form a partition of $\mathcal{S}$. There are $\binom{n}{i}$ choices for the set A in an ordered pair in $\mathcal{S}_i$. Once this set is fixed there are $\binom{i}{2}$ choices for x, y. Therefore

$$|\mathcal{S}_i| = \binom{n}{i}\binom{i}{2},$$

and

$$|\mathcal{S}| = \sum_{i=1}^{n} |\mathcal{S}_i| = \sum_{i=1}^{n} \binom{n}{i}\binom{i}{2}.$$

The result is given by noting that the two expressions for $|\mathcal{S}|$ are equal.

3. A sequence $a_1 a_2 \cdots a_n$ where $a_i \in [m]$ is *increasing* if $a_{i+1} \geq a_i$ for $1 \leq i < n$. Show that the number of such increasing sequences is $\binom{m+n-1}{n}$.

Solution Let x_j be the number of times j appears in the sequence, for $j = 1, \ldots, m$. We have

$$x_j \geq 0 \text{ for } j = 0, 1, \ldots, m \text{ and } x_1 + \cdots + x_m = n. \tag{1}$$

Conversely, given a sequence σ satisfying (1) we get the increasing sequence consisting of x_1 1's followed by x_2 2's etc. The mappings σ to x are inverse to each other. Thus the number of f's is equal to the number of x's satisfying (1), which is $\binom{m+n-1}{n}$.