21-301 Combinatorics
Homework 10
Due: Wednesday, December 6

1. Consider the following game: There is a pile of n chips. A move consists of removing any *proper* factor of n chips from the pile. The player to leave a pile with one chip wins. Determine the N and P positions and a winning strategy from an N position.

   Solution n is a P-position iff it is odd. If n is even then the next player can simply remove one chip. If n is odd, then any factor of n is also odd.

2. Consider the following game: There is a single pile of n chips. A move consists of removing (i) any *even* number of chips provided it is not the whole pile, or (ii) the whole pile, but only if it has 2 (mod 3) chips. The terminal positions are zero and one. Determine the Sprague-Grundy numbers of each pile size.
   (Compute the first 15 numbers and see if you can see see a pattern.)

   Solution The Sprague-Grundy function g is given by

   $$g(0) = g(1) = g(4) = 0 \, and \, g(2) = g(3) = 1 \, and \, g(k) = \lfloor n/2 \rfloor - 1 \, for \, n \geq 5.$$

   We verify the last claim by induction. It can be checked for $n = 5, 6$. Suppose next that $k > 3$. Then if $*$ is $g(0) = 0$ for $n \mod 3 = 2$ and not there otherwise,

   $$\begin{aligned} g(2k) &= mex\{g(2k-2), g(2k-4), \ldots, g(6), g(4), g(2), *\} \\ &= mex\{k-2, k-3, \ldots, 2, 0, 1, *\} \\ &= k-1. \\ g(2k+1) &= mex\{g(2k-1), g(2k-3), \ldots, g(5), g(3), g(1), *\} \\ &= mex\{k-2, k-3, \ldots, 2, 1, 0, *\} \\ &= k-1. \end{aligned}$$

3. Consider the following multi-pile game. A move consists of either (i) removing one, two or three chips from any pile, or (ii) splitting a pile of

size $n \geq 2$ into two piles of sizes 1 and $n-1$. Determine the Sprague-Grundy numbers for a game that starts with a single pile of size n. (Compute the first 15 numbers and see if you can see see a pattern.)

Suppose that the current position consists of three piles of sizes 3,5 and 7. Show that this is an N-position and find all of the winning moves.

Solution

$$g(i) = i,\ i = 0, 1, 2\ and\ g(3) = 4. \tag{1}$$

$$g(4k) = 0,\ g(4k+1) = 3,\ g(4k+2) = 1,\ g(4k+3) = 2\ for\ k \geq 1. \tag{2}$$

To verify (1) remember that moves from $n = 3$ yield 0,1,2 or 1+2 and $g(1+2) = g(1) \oplus g(2)$. So $g(3) = mex\{g(0), g(1), g(2), g(1) \oplus g(2)\} = mex\{0, 1, 2, 1 \oplus 2\} = mex\{0, 1, 2, 3\} = 4$.

We verify (2) by induction. The case $k = 1$ is done case by case:

$$g(4) = mex\{g(1), g(2), g(3), g(3) \oplus g(1)\}$$
$$= mex\{1, 2, 3, 4 \oplus 1\} = mex\{1, 2, 3, 5\} = 0.$$

$$g(5) = mex\{g(2), g(3), g(4), g(4) \oplus g(1)\}$$
$$= mex\{2, 4, 0, 0 \oplus 1\} = mex\{2, 4, 0, 1\} = 3.$$

$$g(6) = mex\{g(3), g(4), g(5), g(5) \oplus g(1)\}$$
$$= mex\{4, 0, 3, 3 \oplus 1\} = mex\{4, 0, 3, 2\} = 1.$$

$$g(7) = mex\{g(4), g(5), g(6), g(6) \oplus g(1)\}$$
$$= mex\{0, 3, 1, 1 \oplus 1\} = mex\{0, 3, 1, 0\} = 2.$$

We verify (2) by induction on k:

$$g(4k) = mex\{g(4k-1), g(4k-2), g(4k-3), g(4k-1) \oplus 1\}$$
$$= mex\{2, 1, 3, 2 \oplus 1\} = mex\{2, 1, 3, 3\} = 0.$$

$$g(4k+1) = mex\{g(4k), g(4k-1), g(4k-2), g(4k) \oplus 1\}$$
$$= mex\{0, 2, 1, 0 \oplus 1\} = mex\{0, 2, 1, 1\} = 3.$$

$$g(4k+2) = mex\{g(4k+1), g(4k), g(4k--1), g(4k+1) \oplus 1\}$$
$$= mex\{3, 0, 2, 3 \oplus 1\} = mex\{3, 0, 2, 2\} = 1.$$

$$g(4k+3) = mex\{g(4k+2), g(4k+1), g(4k), g(4k+2) \oplus 1\}$$
$$= mex\{0, 3, 1, 0 \oplus 1\} = mex\{0, 3, 1, 1\} = 2.$$

If the current position is 3,5,7 then is value is $g(3, 5, 7) = g(3) \oplus g(5) \oplus g(7) = 4 \oplus 3 \oplus 2 = 5 \neq 0$ and so it is an N-position.

The unique winning move is to take 2 chips from the 3-pile. The new position has value $g(1) \oplus g(5) \oplus g(7) = 1 \oplus 3 \oplus 2 = 0$.