Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2006: Test 1

Name:____________________________________

Problem	Points	Score
1	33	
2	33	
3	34	
Total	100	

Q1: (33pts)
(a): Given integers $m, n > 0$ and an integer $a \geq 0$, show that the number of functions f from $[n]$ to $[m]$ which satisfy $f(i+1) \geq f(i) + a$ for $1 \leq i \leq n-1$ is
$$\binom{m+n-a(n-1)-1}{n}.$$
[Hint: Let $x_i = f(i) - f(i-1)$ for $i = 2, 3, \ldots, n$. Define x_1 and x_{n+1} suitably and count the number of choices for $x_1, x_2, \ldots, x_{n+1}$.]
(b): Assuming that $a \geq 1$, use the answer to (a) to find the number of subsets S of $[m]$ which have n elements and satisfy $|x - y| \geq a$ for $x, y \in S$, $x \neq y$.

Q2: (33pts)

(a): We have n boxes $B_1, B_2, \ldots, B_n$ and $2n$ distinguishable balls $b_1, b_2, \ldots, b_{2n}$. Show that there are $\frac{(2n)!}{2^n}$ ways to place the balls into the boxes so that each box gets two balls.

(b): An allocation of balls to boxes is said to be *scrambled* if there does **not** exist i such that box B_i contains balls b_{2i-1}, b_{2i}. Use the Inclusion-Exclusion formula to determine the number of scrambled allocations.

Re-call that if $A_1, A_2, \ldots, A_N \subseteq A$ then

$$\left|\bigcap_{i=1}^{N} \bar{A}_i\right| = \sum_{S\subseteq[N]} (-1)^{|S|}|A_S|.$$

Q3: (34pts) The sequence $a_0, a_1, \ldots, a_n, \ldots$ satisfies the following: $a_0 = 1$ and

$$a_n - 3a_{n-1} = 1$$

for $n \geq 1$.

(a): Find the generating function $a(x) = \sum_{n=0}^{\infty} a_n x^n$.

(b): Find an expression for a_n, $n \geq 0$.