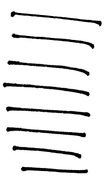
12\01\06



A move: (i) Remove an even number but not everything

(ii) Whole pile, if it is odd

n	0	1	2	3	4	5	6	7	8	9	10	
$g(n)$	0	1	0	2	1	3	2	4	3	5	4	..

$$g(2k) = k-1 \qquad k \geq 1$$

$$g(2k-1) = k \qquad k \geq 1$$

<u>Proof by induction on k.</u>

Small k are done. Suppose $k > 1$

$$g(2k) = \text{mex}\{\, g(2k-2), \; \ldots, g(2) \,\} = k-1$$

$$k-2, \; k-3, \ldots, \; 0 \qquad \text{Induction}$$

$$g(2k-1) = \text{mex}\{\, g(2k-3), g(2k-5), \ldots, g(1), g(0) \,\} = k$$

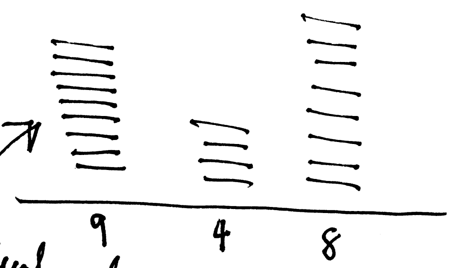
$$k-1, \; k-2 \qquad , \; 1, \; 0 \qquad \text{induction.}$$

Theorem

$g_1, \dots, g_n$ are SG functions of games $1, 2, \dots, p$ then if g is SG for the sum of those games, then

$$g(x) = g_1(x_1) \oplus g_2(x_2) \oplus \cdots \oplus g_p(x_p)$$

Nim: $\underline{g_i(x_i) = x_i}$

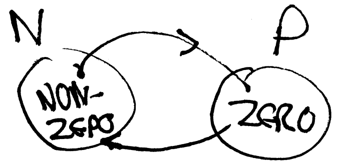


Individual pile (game)

Grundy numbers for a one pile game of Nim

n	0	1	2	3	4	5	...
$g(n)$	0	1	2	3			

$$\underbrace{g(9,4,8)}_{\text{N position}} = \begin{array}{c} 1000 \\ 0100 \\ 1000 \\ \hline 0101 \end{array} \neq 0 \quad \longrightarrow \quad \begin{array}{c} 1001 \\ 0001 \\ 1000 \\ \hline 0000 \end{array}$$



<u>Proof</u> $X = \{ (x_1, x_2, \dots, x_p) \}$
$\{ \text{posistions} \}$

(a) · if $x \in X$ and $g(x) = b > a$ then
$\exists\, x' \in N^+(x)$ s.t. $g(x') = a$.

(b) $x \in X$ & $g(x) = b$ and $x' \in N^+(x) \Rightarrow g(x') \neq g(x)$.

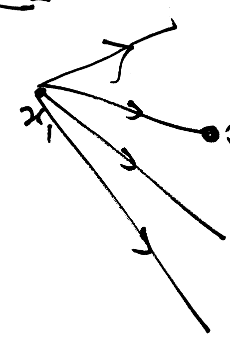
(a) $b \longrightarrow a$ Define $d = a \oplus b$

$a = d \oplus b$

$= d \oplus g(x_1) \oplus g(x_2) \oplus \dots \oplus g(x_p)$

Suppose $d \oplus g(x_1) < g(x_1)$ — $\boxed{\text{prove } \exists i : d \oplus g(x_i) < g(x_i)}$

<u>Game 1</u>



x_1' s.t. $g_1(x_1') = d \oplus g(x_1)$

$a = g_1(x_1') \oplus g_2(x_2) \oplus \dots \oplus g_p(x_p)$

Suppose

$$d = \cdots 0001\overset{d_k=1}{} \underbrace{* * * * * * * * *}_{k^{th} \text{ position}}$$

$$d_k = a_k \oplus b_k$$

$a_k = 0$, $b_k = 1$

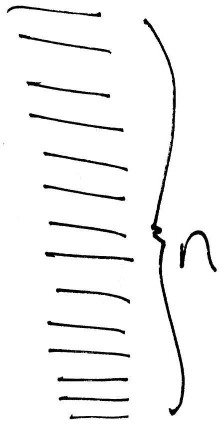
sum of 1's

$d = a \oplus b$

$a < b$

So $\exists i$: $g_i(x_i)$ has a 1 in position k

$$d \oplus g_i(x_i) < g_i(x_i)$$



One pile: First move: Remove up to $n-1$ chips

Subsequent moves: suppose last move removed x chips. Can remove up to x chips.

<u>Position</u>

(n, x)

#chips #previously taken.

n = 0 1 2 3 4 5 6 7 8 ...

P P P

(1,1)
N

(0,0) P (1,0) P (2,1) P (2,2) N

Starting Position

(3,1) N (3,2) N (3,3) N

(4,1) P (4,2) P (4,3) P (4,4) N

Start Positions

0	1	2	3	4	5	6	7	8
P	P	P	N	P	N	N	N	P

Losing positions at beginning are 2^k, $k \geq 0$

Winning Strategy

Suppose I see (n, x)

$x \geq 8$

1 1 0 1 1 0 1 0 0 0 0 0

Remove