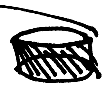
11\27\06

Combinatorial Games

Game 1 Take-Away Game

Start with n chips  poker?

Players alternate.

A move consists of taking 1, 2, 3 or 4 chips.

Winner takes the last chip(s). / You lose if you cannot make a move.

How to play? A goes first, B goes 2nd

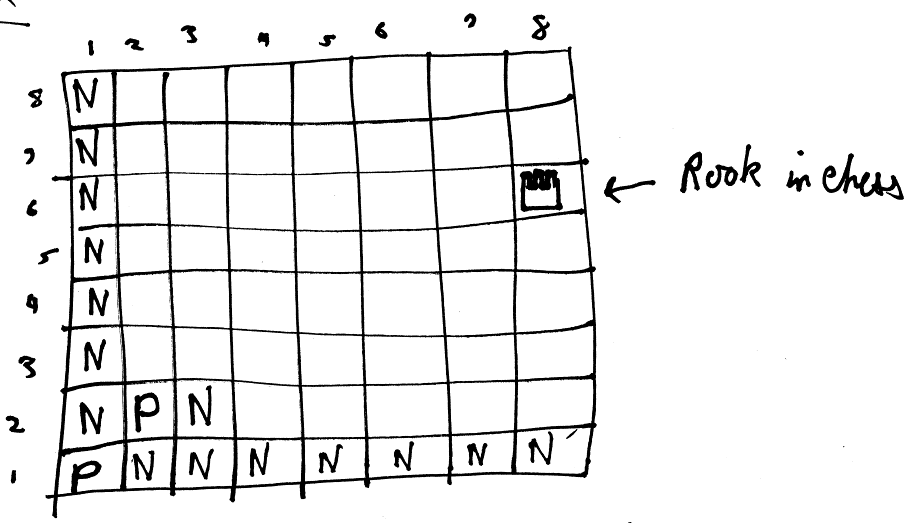
	P	N	N	N	N	P	N	N	N	
n	0	1	2	3	4	5	6	7	8	..
Winner	B	A	A	A	A	B	A	A	A	...

pile of chips

Two sorts of position: N position – next player wins
P position – previous player wins

Game 2

2-pile NIM



A more "pushe" Rook close to (1,1) – legally.

What are the N positions

What are the P positions – (i,i) diagonal

Game 3

Replace rook by a queen

(Wythoff's Nim)

["Last year in Marienbad" – There is a game of
"French" movie 3-pile nim]

Game 3 — Geography

$W = \{\text{words}\}$ could be all the countries in world

Construct sequence

$$\begin{array}{ccccccc} A & B & A & B \cdots \\ w_0, & w_1, & w_2, & \cdots & w_i, & w_{i+1} & \cdots \end{array}$$

$w_i \in W$ and first letter of w_{i+1} = last letter of w_i

BRAZIL, LAPLAND, DENMARK, ...

Winner is last person to name a country

- no repeats.

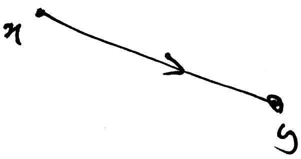
[A similar game can be solved via matchings in graphs]

Mathematical Abstraction

We have digraph $D = (X, A)$

A vertex for each possible position.

An arc



if one can move from position x to position y

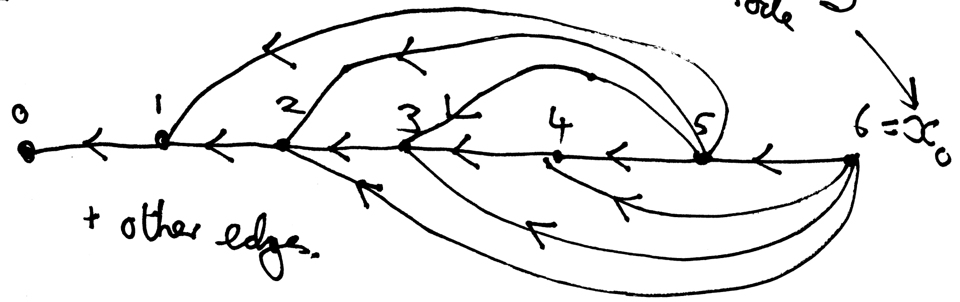
[No directed cycles]

Game 1

$n = 6$

starting node

0 is a sink



+ other edges.

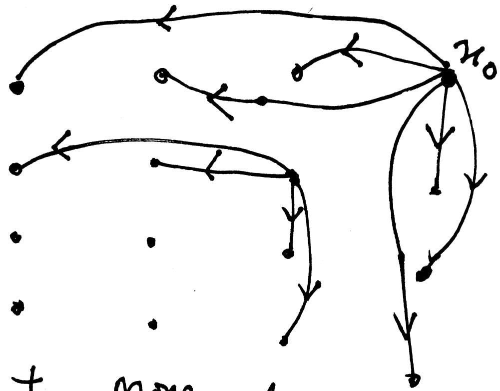
Put a token on x_0. Players alternately move token one edge at a time.

Game ends when token is moved to a Sink

A sink is a vertex of <u>out-degree</u> zero

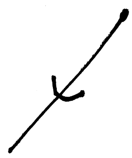
<u>Example 2</u>

$n = 4$



+ more edges.

Example 2a: add some arcs



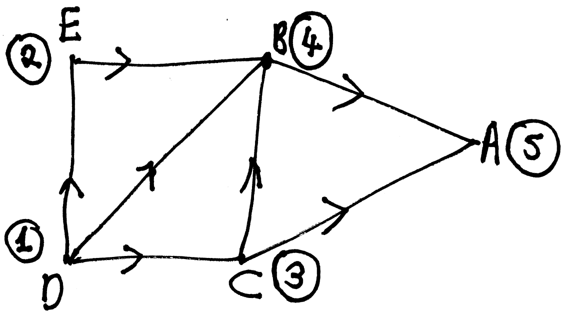
We must argue that game will finish.

Topological Numbering

$D = (X, A) \qquad |X| = n$

A topological number $f : X \longrightarrow [n]$

s.t. $\quad (x, y) \in A \Rightarrow f(x) < f(y)$

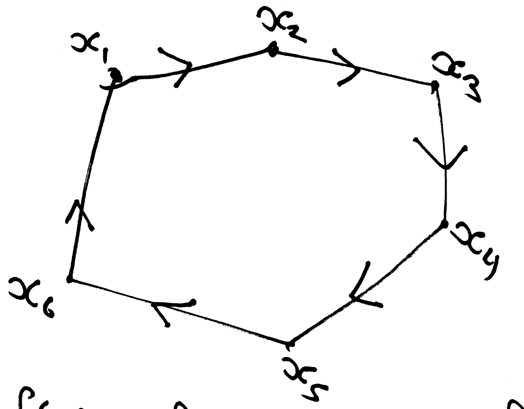


NO DIRECTED CYCLES

Thm ~~Every~~ ~~a~~ A finite digraph is acyclic IFF IT HAS A TOPOLOGICAL NUMBERING.

(a) Suppose $\exists$ a top. num. ing.

$\Rightarrow$ No directed cycles.



$f(x_1) < f(x_2) < f(x_3) < f(x_4) < f(x_5) < f(x_6) < f(x_1)$

contradiction.

$\Leftarrow$ Assume no directed cycle.

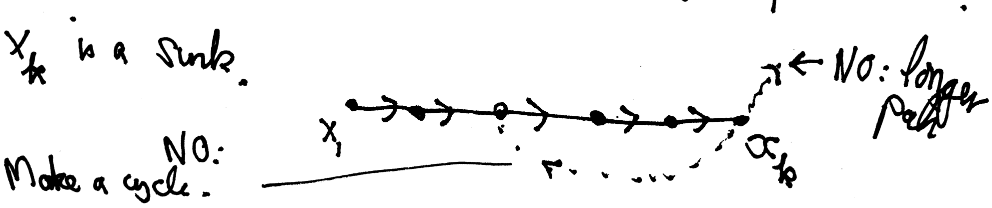
We construct numbering by induction on $|X|$.

$|X| = 1$: trivial

Claim: D has at least one sink.

Let $P = (x_1, x_2, \ldots, x_k)$ be a longest path in D.

$\Rightarrow$ x_k is a sink.



To number D.

1: Choose a sink s:

$f(s) = n$

2: number $D \setminus \{s\}$ — induction

$f : X \longrightarrow [n]$ is a topological number.

$(x,y) \in A \ni f(x) < f(y)$

$x, y \neq s$

$y = s$: $f(y) = n$ $\checkmark$