11\22\06

"Dual" Dilworth Theorem:

If m = max. length of chain, then P can be covered by m anti-chains.

[You need at least m anti-chains because
$|\text{Chain} \cap \text{Antichain}| \leq 1$]

A is the set of <u>maximal</u> elements of P.

x is maximal if $\nexists\, y \in P$ s.t. $y > x$.

<u>A is an anti-chain</u>

Proof goes by induction on $|P|$

$$P' = P / A$$

Max. length of a chain in P' is $\leq m-1$.

Suppose $\exists$ a chain in P' of length m

$$x_1 < x_2 < \cdots < x_m$$

NOT MAXIMAL, NOT IN A.

$$x_m < y$$

$\Rightarrow \exists$ chain of length $m+1$ in P — contradiction

By induction $P' = A_1 \cup A_2 \cup \cdots \cup A_{m-1}$

all anti-chains

$$P = A_1 \cup A_2 \cup \cdots \cup A_{m-1} \cup A.$$

Erdös-Szekerés from Dilworth

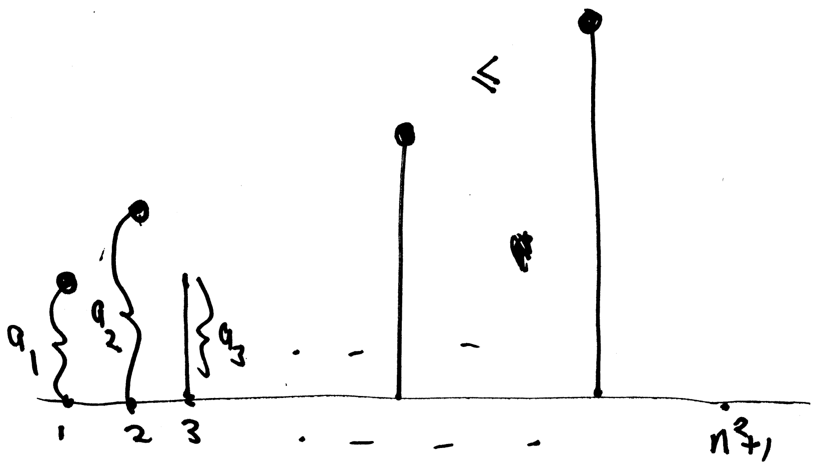
Given $a_1, a_2, \ldots, a_{n^2+1}$

Let $P = \{(i, a_i) : 1 \leq i \leq n^2+1\}$

and say

$$(i, a_i) < (j, a_j)$$

if $i < j$ and $a_i \leq a_j$



Apply Dilworth:

Suppose there does not exist a chain of length $n+1$

[A chain of length $n+1$ corresponds to a monotone increasing of length $n+1$]

$\Rightarrow$ You need at least $n+1$ chains to cover P

Dilworth $\Rightarrow$ $\exists$ anti-chain of size $n+1$

(otherwise $\Rightarrow$ we only need $\leq n$ chains]

Suppose

$$A = \{(i_1, a_{i_1}), (i_2, a_{i_2}), \dots (i_{n+1}, a_{i_{n+1}})\}$$

is an anti-chain, where $i_1 < i_2 < \dots < i_{n+1}$.

Then

$$a_{i_{t+1}} < a_{i_t} \Rightarrow$$ A define a monotone decreasing sequence of length $n+1$.

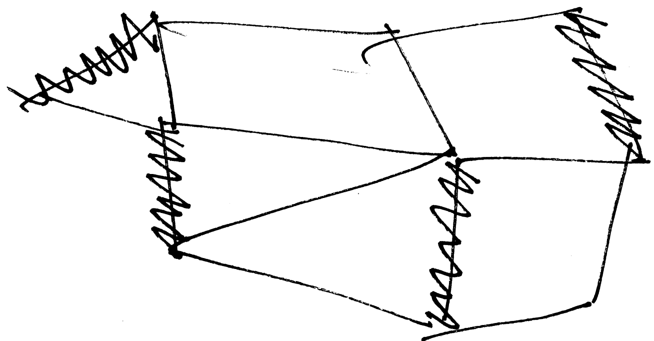
else

$$(i_t, a_{i_t}) < (i_{t+1}, a_{i_{t+1}})$$

i.e. A would not be an anti-chain.

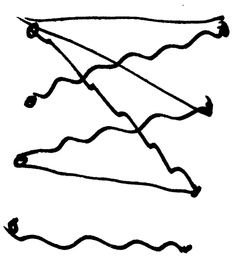
Hall's Theorem

Matchings in bipartite graphs.



A matching is a set of disjoint edges

A matching is perfect if it covers all vertices.

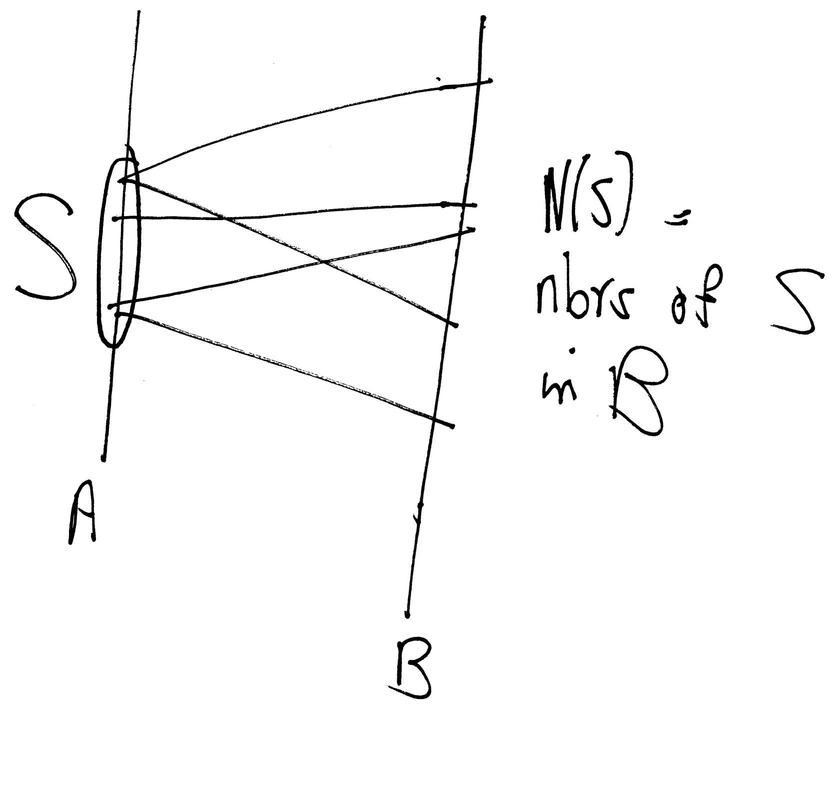


$G = (\underbrace{A, B}_{\text{Bipartition of vertex}}, E)$ is a bipartite graph

G has a matching of size $|A|$ iff

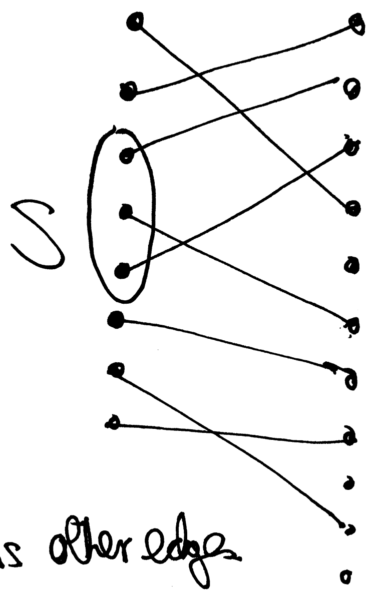
$|N(S)| \geq |S|$ for all $S \subseteq A$

Hall's Condition



iff

Suppose first that G has a matching of size $|A|$



$\forall S$ has $\geq |S|$ nbrs

($|S|$ from matching).

plus other edges

Hall condition then implies that

$$|N(\{a_1, \dots, a_h\})| \geq h$$

and so

$$|B| \geq h+k = s$$

Apply Dilworth

P is the union of s chains

A chain consist of a, b or $\{a,b\}$ where (a,b) is an edge.

These s chains can be split up as

(i) a matching M of size m

(ii) $|A| - m$ members of A

(iii) $|B| - m$ members of B.

Hall gave us

$$m + (|A|-m) + (|B|-m) = s \leq |B|$$

$$\Downarrow$$

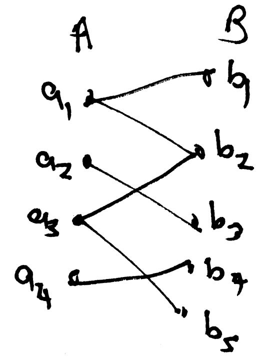
$$m \geq |A|.$$

$m \leq |A|$

We use Dilworth's Theorem to prove converse:

$$P = A \cup B$$

$a \in A,\ b \in B \qquad a < b$ if (a,b) is an edge.



$$a_3 < b_2$$

A is an anti-chain
B is an anti-chain

Largest anti-chain is X

$$X = \{\underbrace{a_1, a_2, \ldots, a_h}_{\in A},\ \underbrace{b_1, b_2, \ldots, b_k}_{\in B}\}$$

$s = h + k$

$$N(\{a_1, a_2, \ldots, a_h\}) \subseteq B \setminus \{b_1, b_2, \ldots, b_k\}$$

otherwise X is not an anti-chain.

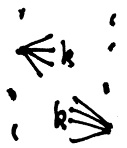
Marriage Theorem

$G = (A \cup B, E)$ is a k-regular $(k \geq 1)$ bipartite graph [Regular: every vertex is of degree k]

~~Then~~ G has a perfect matching.

Proof

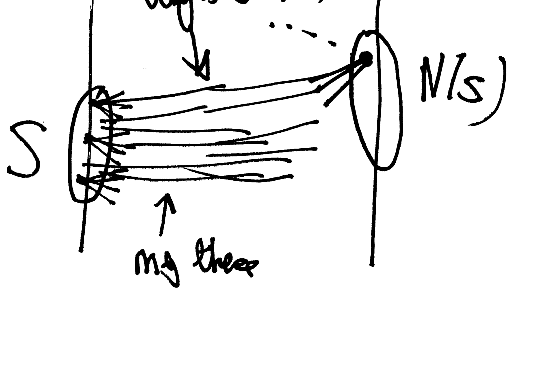
#edges in G is $k|A| = k|B| \Rightarrow |A| = |B|$.



We need to show $\exists$ matching of size $|A|$.

We need to show

$$|N(S)| \geq |S| \quad \forall\, S \subseteq A.$$



$k|S| = m$
$\leq k|N(S)|$