11\11\06

POSETS (Partially Ordered Sets)

A poset $(P, \leqslant)$

P: set

$\leqslant$: partial order on P

$\leqslant$ is a relation.

(i) $a \leqslant a \quad \forall a \in P$

(ii) $a \leqslant b$ & $b \leqslant a \Rightarrow a = b$

(iii) $a \leqslant b$ & $b \leqslant c \Rightarrow a \leqslant c$

[We write $a < b$ if $a \leqslant b$ and $a \neq b$]

Examples

(i) $P = \{1, 2, 3, \ldots\}$ and $\leqslant$ has usual meaning

(ii) $P = \{1, 2, 3, \ldots\}$ and $a \leqslant b$ iff a is a factor of b.

(iii) $P = \{A_1, A_2, \ldots, A_m\}$ where $A_1, \ldots, A_m$ are sets.

$|P| = m$

$A \leqslant B$ iff $A \subseteq B$

Notation
a & b are comparable if $a \leqslant b$ or $b \leqslant a$.

A poset without incomparable elements is called linear or a total order. $\leqslant$ in ordinary sense.

[Totally ordered set = Poset where $\leqslant$ is total.]

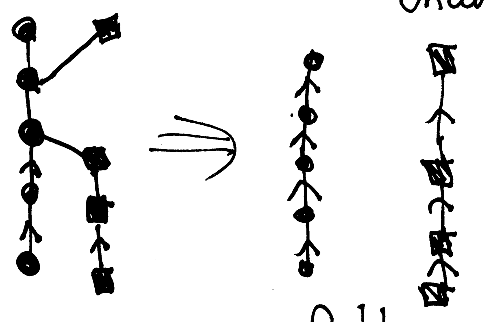
A chain is a sequence $a_1 < a_2 < \cdots < a_n$

An anti-chain is a set for which every pair of elements is incomparable.

A Sperner family is an anti-chain in example (iii).

Suppose $P = C_1 \cup C_2 \cup \cdots \cup C_m$ [Partition]

chains



Partition

A is an anti-chain $\Rightarrow |A| \leq m$

[If $|A| \geq m+1$ then by PHP there exists i s.t. $|A \cap C_i| \geq 2$ – contradiction]

$\max |A| \leq \min P = C_1 \cup C_2 \cup \cdots C_m$

A is an anti-chain

m

Chain decomposition

Dilworth: $|P| < \infty$, $=$

We show that if μ is the size of the largest anti-chain, then $\exists$ a chain decomposition $C_1, C_2, \ldots, C_\mu$.

<u>Proof is by induction</u> on $|P|$

Base Case: $|P| \leq 1$

Now assume $|P| > 1$ and μ = maximum size of an anti-chain.

$C = x_1 < x_2 < \cdots < x_p$ be a <u>maximal</u> chain
i.e. cannot add elements.

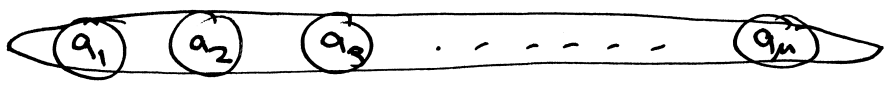
Look at $P \setminus C$

Case 1: Max. anti-chain size in $P \setminus C$ is $\leq \mu - 1$
By induction, $P \setminus C = C_1 \cup C_2 \cup \cdots C_{\mu-1}$
so $P = C_1 \cup C_2 \cup \cdots \cup C_{\mu-1} \cup C$ ✓

<u>Case 2</u> $P \setminus C$ contains an anti-chain

$$A = \{a_1, a_2, \ldots, a_\mu\}$$

$$P^+ = \{x \in P : x \geq a_i, \text{for some } i\} \supseteq A$$

A 

$$P^- = \{x \in P : x \leq a_i, \text{for some } i\} \supseteq A$$

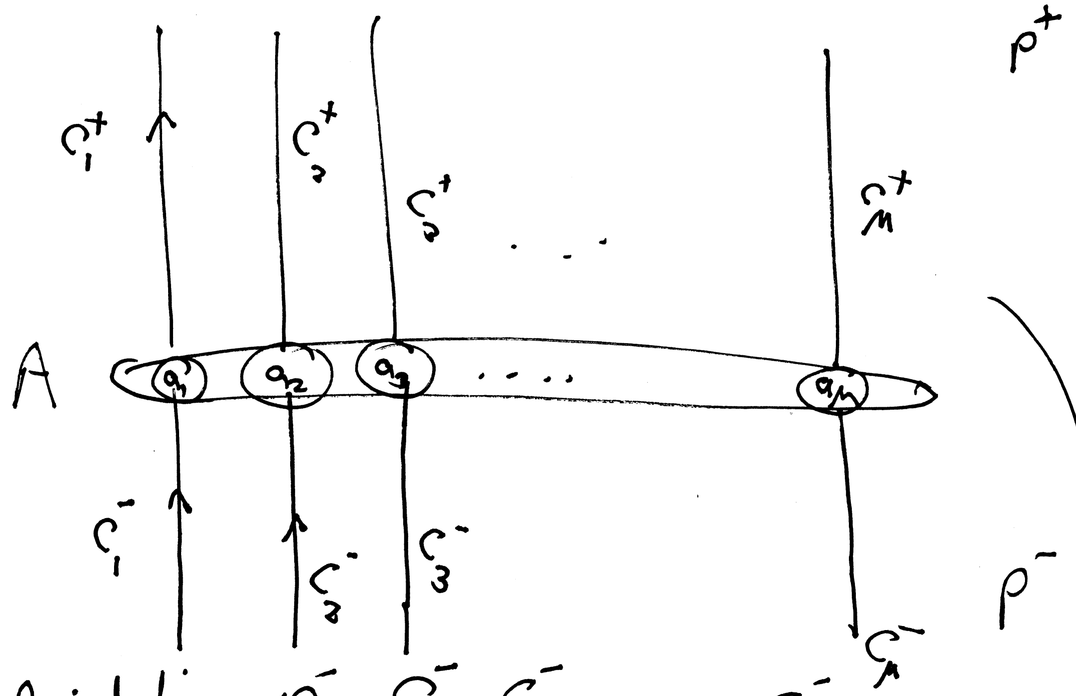
(i) $P = P^- \cup A \cup P^+$ — else I can make A bigger

(ii) $P^- \cap P^+ = A$ — else $\exists\ a_i \leq x < a_j \Rightarrow a_i \,\&\, a_j$ are comparable

(iii) $x_p \notin P^-$ — else C can be extended by something in A

$|P^-| < |P|$ $[A \cap C = \emptyset]$

& $|P^+| < |P|$



By induction, $P^- = C_1^- \cup C_2^- \cup \dots \cup C_\mu^-$

The a_i's go one to a chain.

$a_1 \in C_1^- \longrightarrow a_1$

Is a partition into μ chains.