11\08

$R(k,k) > 2^{k/2}$

$n = 2^{k/2}$ and randomly color edges of K_n.

With positive probabilty there is no Red or Blue k-clique.

so $\exists$ a 2-coloring of K_n without a mono-chromatic k-clique.

So $R(k,k) > n = 2^{k/2}$.

$\mathcal{E}_R$: event, $\exists$ Red k-clique

$\mathcal{E}_B$: ... .. Blue

We show

$$P_r(\mathcal{E}_R \cup \mathcal{E}_B) < 1$$

$\Pr(\mathcal{E}_R)$ is hard to compute exactly.

Let $C_1, C_2, \ldots, C_N$, $N = \binom{n}{k}$ be a list of all k-sets in $[n]$

$\mathcal{E}_{Rj}$ = event C_j is Red.

$$\mathcal{E}_R = \bigcup_{j=1}^{N} \mathcal{E}_{Rj}$$

$$
\begin{aligned}
Pr(\mathcal{E}_R \cup \mathcal{E}_B) &\leq Pr(\mathcal{E}_R) + Pr(\mathcal{E}_B) \\
&= 2Pr(\mathcal{E}_R) \\
&= 2Pr\left(\bigcup_{j=1}^{N} \mathcal{E}_{R_j}\right) \\
&\leq 2\sum_{j=1}^{N} Pr(\mathcal{E}_{R_j}) \\
&= 2\sum_{j=1}^{N} \left(\frac{1}{2}\right)^{\binom{k}{2}} \\
&= 2N\left(\frac{1}{2}\right)^{\binom{k}{2}}
\end{aligned}
$$

$$= \binom{n}{k}\left(\frac{1}{2}\right)^{\binom{k}{2}} - 1$$

$$\leq \frac{n^k}{k!}\left(\frac{1}{2}\right)^{\binom{k}{2}} - 1$$

$$= \frac{2^{k^2/2}}{k!}\left(\frac{1}{2}\right)^{\binom{k}{2}} - 1$$

$$= \frac{2^{k^2/2 - \frac{k^2}{2} + \frac{k}{2} + 1}}{k!}$$

$$= \frac{2^{1+k/2}}{k!} < 1 \text{ for } k \geq 5.$$

Ramsey Theorem in General

Given $r, s \geq 1$ and

$q_1, q_2, \ldots, q_s$ $\boxed{s \text{ colors}}$

there exists

$N(q_1, q_2, \ldots, q_s; r)$ such

that:

In any s-coloring of the r-sets of $[n]$

$n=4$
$r=3$
$s=2$

$\{1,2,3\} \leftarrow R$
$\{1,2,4\} \leftarrow R$
$\{1,3,4\} \leftarrow B$
$\{2,3,4\} \leftarrow R$

2^4 colorings

$\exists\, i$ and a set T of size q_i such that all sets in $\binom{T}{r}$ have color i.

We have done

$r = s = 2$

$q_1 = k$

$q_2 = l$

Proof is by induction on r.

$r=1$: Pigeon-hole

$r=2$: We're done this

Assume now that $r \geqslant 3$.

n is very large.

Υ is a 2-coloring of $\binom{[n]]}{r}$

1 $\bullet$

All $(r-1)$-sets in $\{2, \dots n\}$

Υ induces a coloring σ of

$$\sigma(S) = \Upsilon(S \cup \{1\}).$$

We start with $s=2$.

Induction hypothesis

(a) $N(q_1, q_2 : r-1)$ exists for all q_1, q_2.

(b) $N(k, l : r)$ exists for $k+l < q_1 + q_2$.

$N(r, r : r) = r$. $\qquad$ $q_1, q_2 \geq r$

n is large enough*
that either
(i) $\exists$ large A such that
in σ all $(r-1)$-sets are Red
or
(ii) $\exists$ large B such in σ
all $(r-1)$-sets are Blue.

[By induction]

Assume (i) and A is large* enough so that under T either (a) there is a set A_1 of size $q_1 - 1$ with all r-sets Red or

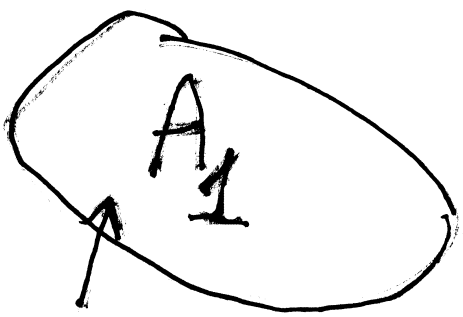
(b) $\exists$ a set B_1 of size q_2 with all r-sets blue.

*Induction: $q_1 - 1 + q_2 < q_1 + q_2$

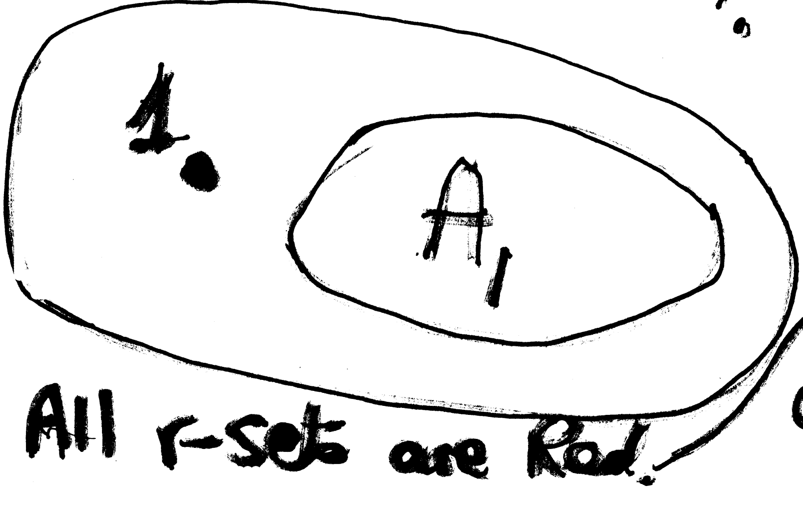
(b) holds — DONE

(a) - holds

$|A_1| = q_1 - 1$



all r-sets are Red under γ_1.



All (r-1)-sets in A_1 are Red in σ.

All r-sets are Red.