11\06\2006

Choose 6 people.

Either $\exists$ "3 people who know each other"

or $\exists$ "3 people who do not know each other"

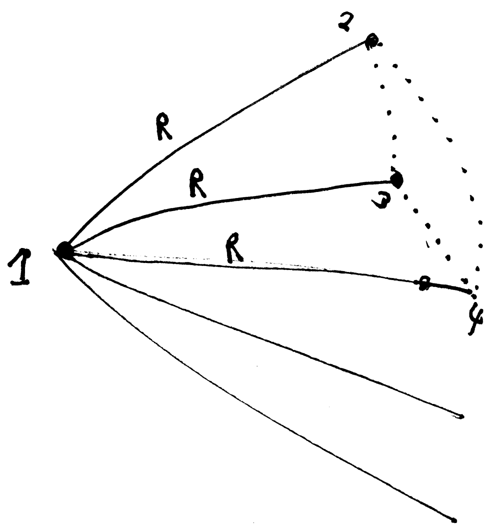
or both can happen.

Mathematically

Color the edges of K_6 Red or Blue.

$\exists$ a mono-chromatic triangle

i.e. a Δ with edges of one color.



5 edges colored R&B

Pigeon-hole $\Rightarrow \exists$ a color used ≥ 3 times.

Generalisation: Ramsey's Theorem

Given k, l, $\exists$ integer $R(k,l)$ such that if $n \geq R(k,l)$ and the edges of K_n are 2-colored, then either (i) $\exists$ a Red K_k

or (ii) $\exists$ a Blue K_l (inclusive ~~or~~)

What we have shown is that $R(3,3) = 6$

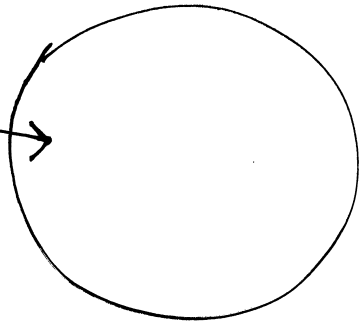
What is known

$$R(1, k) = 1$$

$$R(2, k) = k$$

$$R(k, 2) = k$$

K_k



(i) $\exists$ a red edge $\Rightarrow$ Red K_2

(ii) $\nexists$ a red edge $\Rightarrow$ Blue K_k

$R(3,3) = 6$

$R(4,4) = 18$

$R(4,5)$ is known

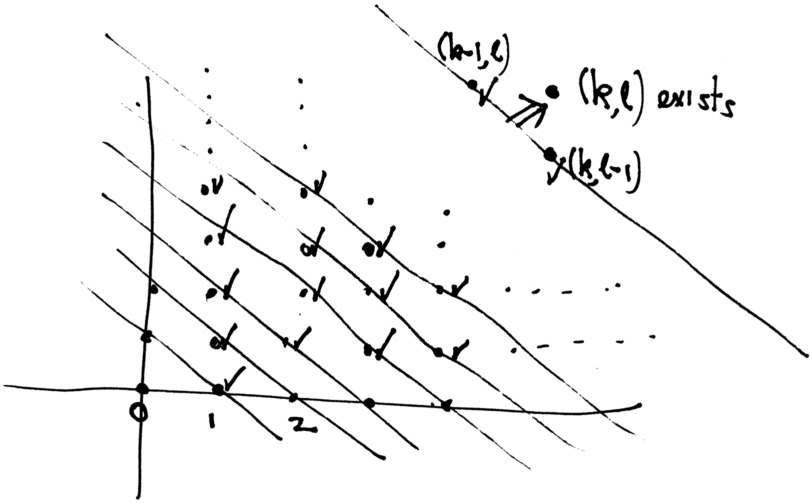
$R(5,5) = ?$

$R(5,5) > 20$

2-colorings of K_{21} is $2^{\binom{21}{2}}$ ← very large.

Proof of existence

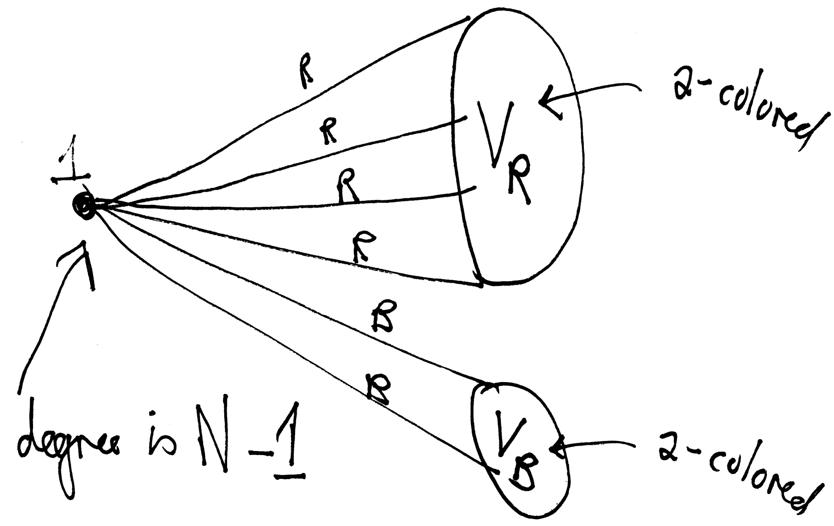
$$R(k,l) \leq R(k,l-1) + R(k-1,l)$$



Existence proof by induction on the sum $k+l$.

$$N = R(k, l-1) + R(k-1, l)$$

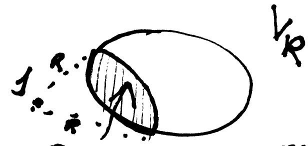
assumed to exist, by induction.



$$|V_R| + |V_B| = N - 1$$

So

Either (i) $|V_R| \geq R(k-1, l)$



Red K_{k-1} or Blue K_l ✓

or (ii) $|V_B| \geq R(k, l-1)$ — similar.

$$R(k,l) \leq R(k,l-1) + R(k-1,l)$$

$$R(k,l) \leq \binom{k+l-2}{k-1}$$ Bounds size of $R(k,l)$

Proof

Induction on $k+l$.

Check small cases $k+l \leq 5$

$$R(k,l) \leq R(k,l-1) + R(k-1,l)$$ just proved

$$\leq \binom{k+l-3}{k-1} + \binom{k+l-3}{k-2}$$ induction

$$= \binom{k+l-2}{k-1}$$

Pascal Δ Identity

$$R(k,k) \leq \binom{2k-2}{k-1} < 4^k$$

$$\underset{\sim\sim}{\sqrt{2}} \overset{?}{<} R(k,k)^{1/k} < 4$$

? Does $\lim_{k\to\infty} \underbrace{R(k,k)^{1/k}}$ exist ?

$$R(k,k) > 2^{k/2}$$

Let $n = \lfloor 2^{k/2} \rfloor \Rightarrow \exists$ a 2-coloring of K_n

without a Red or Blue K_k

How to prove? (i) Construct such a coloring – HARD??

(ii) Probabilistic Method

i.e. Color randomly.