Simple proof

Average size of $|f^{-1}(i)|$ is $\frac{m}{n}$

Since $\sum |f^{-1}(i)| = m$ and we have n tables,

So there has to be at least one i s.t.

$$|f^{-1}(i)| \geq \frac{m}{n}$$

Note that $|f^{-1}(i)|$ is an integer. So

$|f^{-1}(i)| \geq$ "smallest integer, greater than $\frac{m}{n}$" $= \lceil \frac{m}{n} \rceil$

$\boxtimes$

Most common use: is when $m > n$ then we have that there is an i s.t. $|f^{-1}(i)| \geq \lceil \frac{m}{n} \rceil \geq 2$

We have a hole
$\{i, i+1\}$ s.t. i and $i+1 \in A$

Example
A lossless compression cannot compress all files.

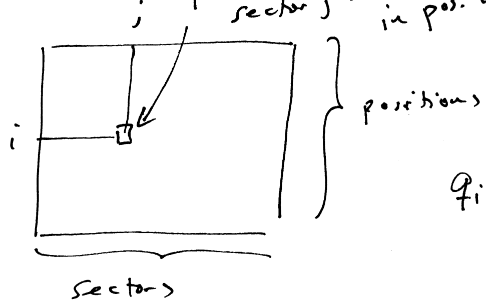
Proof
A (lossless) compression is a map $f : [2^n] \to [2^n]$
if it compresses every file (by at least one bit) then
we can write $f : [2^n] \to [2^{n-1}]$
but then there exists $i \in [2^{n-1}]$ s.t.
$$|f^{-1}(i)| \geq \left\lceil \frac{2^n}{2^{n-1}} \right\rceil = 2$$
So f cannot have an inverse, i.e. it is not lossless. ∎

q_i : number of matches when Disk 2 is in position i.

Disk 2

Sector j will match in 100 position

j put 1 here if sector j is matched in pos. i, 0 o.w.



q_i : sum of row i.

$$\underbrace{q_1 + \dots + q_{200}}_{\text{Sum over all of the matrix.}} = \sum_{\substack{\text{sectors} \\ j}} \#\text{of 1's in column } j = \sum_{\substack{\text{sectors} \\ j}} 100 = 100\cdot$$

avg size of q_i $= \frac{100 \cdot 200}{200} = 100$

So there is a position i s.t. the ~~size~~ $q_i \geq 100$

A monotone subsequence is a subsequence

$a_{i_1}, a_{i_2}, \dots, a_{i_j}$ when $i_1 < i_2 < i_3 < \dots < i_j$

monotone if its increasing or decreasing.

Let $(a_i, a_i^1, \dots, a_i^{\ell-1})$ be the longest <u>increasing</u> subseq. that starts at a_i and $\ell(a_i)$ is its length.

If $\ell(a_i) \geq k+1$ for some i then we're done.

So assume $\ell(a_i) \leq k$ for $i = 1, \dots, k^2+1$

Look at $\ell(a_i) : [k^2+1] \to [k]$

By PP there is an ℓ^* s.t. there are $\left\lceil \frac{k^2+1}{k} \right\rceil = \left\lceil k + \frac{1}{k} \right\rceil = k+$

Call them $i_1, \dots, i_{k+1}$ look at i_1 and i_2

$$\underset{i_1}{\sqcup} \overset{?\,no}{\leq} \underbrace{\underset{i_2}{\sqcup} \leq \dots \leq \dots \leq \sqcup \dots \leq \sqcup}_{\ell^* \text{ places}}$$

So then $a_{i_1} \geq a_{i_2}$

Generalize this to see

$a_{i_1} \geq a_{i_2} \geq a_{i_3} \dots \geq a_{i_k} \geq a_{i_{k+1}}$

decr. subseq. of size $k+1$. $\boxtimes$