10/27/06

G is connected.

Every vertex is even

$\Longrightarrow$ G has an Euler cycle.

<u>Proof</u>

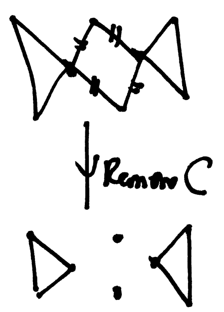
Induction on $|E|$.

$|E| \leq 3$: trivial.

So assume $|E| > 3$

(i) G has at least one cycle C, say.

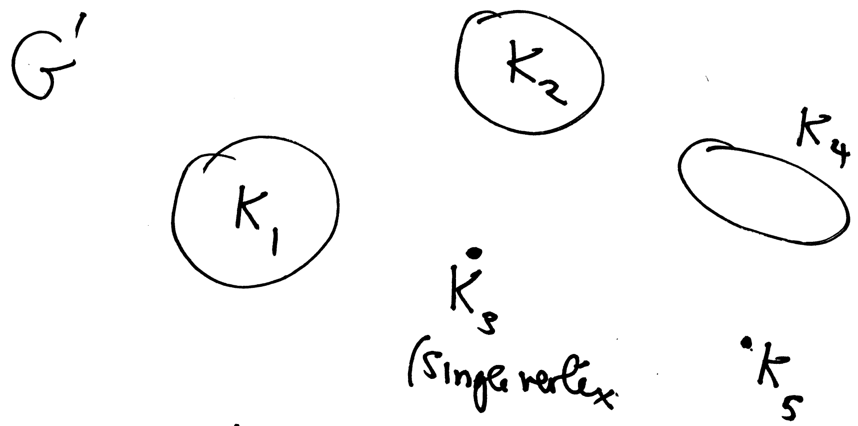
$G' = G \setminus \underset{\substack{\uparrow \\ \text{edges}}}{C} = (V, E')$



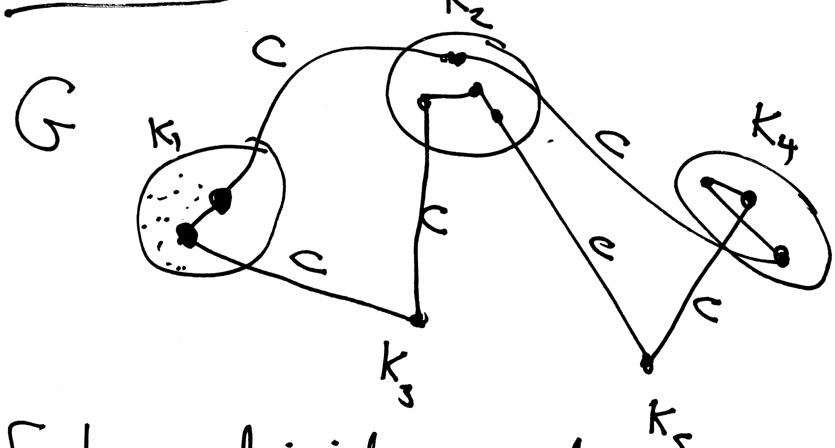
G' has components $K_1, K_2, \ldots, K_r$

Each K_i is connected and has even degrees.

$$\deg_{G'}(v) = \begin{cases} \deg_G(v) & \text{or} \\ \deg_G(v) - 2 & \leftarrow v \in C \end{cases}$$



Add C back

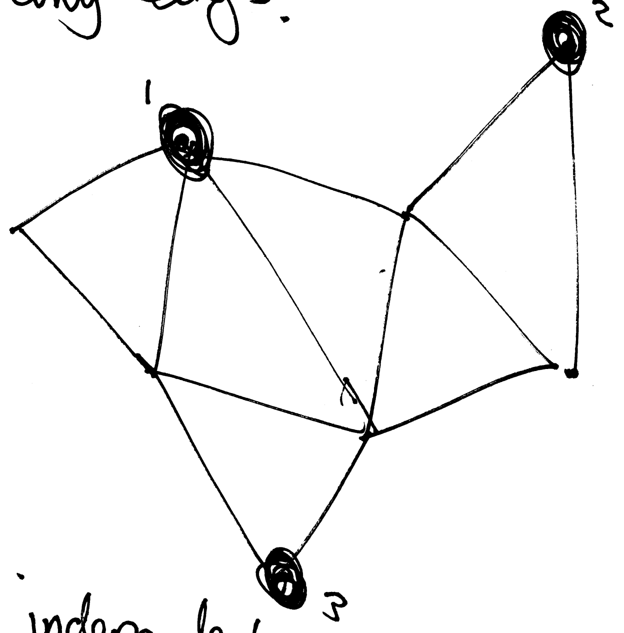


Each non-trivial component has an euler cycle – induction.

Build an euler tour out of C and these [illegible].

Independent Sets

A set of vertices in a graph G is **independent** if it does not contain any edge.



1, 2, 3 are independent

$\alpha(G)$ = size of largest independent set in G.

Turan's Theorem

$$\alpha(G) \geq \frac{n}{\frac{2m}{n} + 1}$$

$$= \frac{n}{\text{average degree} + 1}$$

Observation:

$$\alpha(G) \geq \frac{n}{\text{maximum degree} + 1} \quad \text{is easy.}$$

$\Delta \nearrow$ maximum degree

Greedy Algorithm.

v_1 $\quad$ v_2 repeat

$v_1 \to$ Indep. Set

nbrs deleted

Lost $\leq \Delta + 1$ vertices

$v_1, v_2, \dots, v_k$

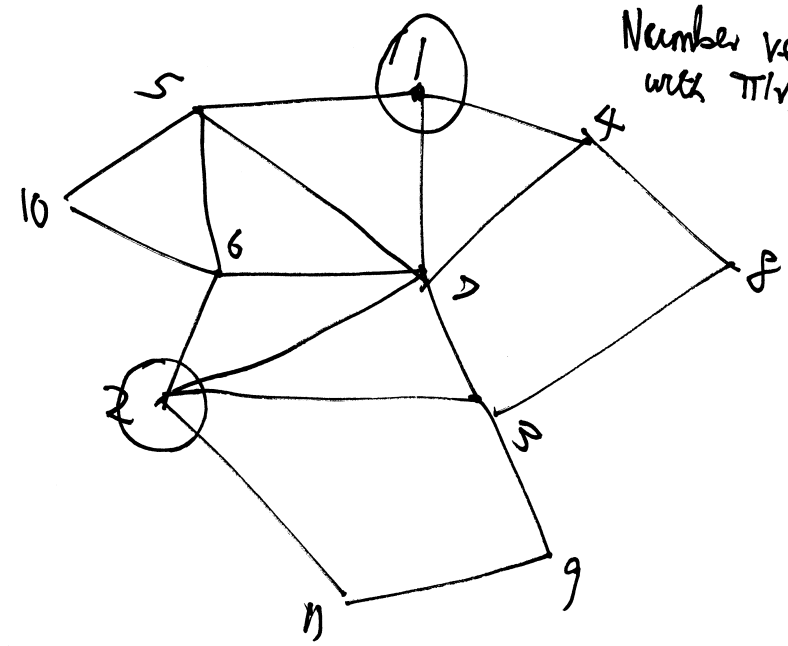
Can keep going as long as $k(\Delta+1) < n$

(i) Let π be a permutation of V.

We will construct an independent set from π.

$$I(\pi) = \{ v : \pi(w) > \pi(v) \text{ for all } w \in N(v) \}$$

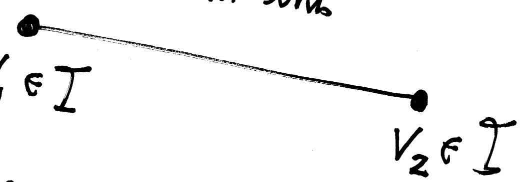
Number vertices with $\pi(v)$



$I(\pi) =$ the two $\odot$

$I(\pi)$ is independent

Suppose we had this



$\pi(v_1) < \pi(v_2) < \pi(v_1)$??

Now suppose that π is a random permutation.

Claim

$$E(|I|) = \sum_{v \in V} \frac{1}{d(v)+1}$$

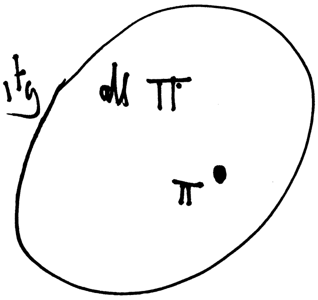
$I = I(\pi)$

π is random

$|\Omega| = n!$

$n = |V|$

Probability space Ω



Random variable

function $\Omega \to \mathbb{R}$

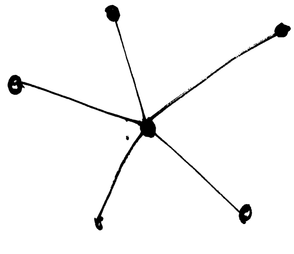
$\pi \longrightarrow |I(\pi)|$

$$\delta(v) = \begin{cases} 1 & v \in I \\ 0 & v \notin I \end{cases}$$

$$|I| = \sum_{v \in V} \delta(v)$$

$$E(|I|) = \sum_{v \in V} E(\delta(v))$$

$$= \sum_{v \in V} \Pr(\delta(v) = 1) = \sum_{v \in V} \frac{1}{d(v)+1}$$



What is probability that $\pi(v)$ is the smallest of those $d(v)+1$ number?

$$\frac{1}{d(v)+1}$$

$\square$

$$E(|I|) = \sum_{v \in V} \frac{1}{d(v)+1}$$

$\exists$ a π such that $|I(\pi)| \geq$

$$\alpha(G) \geq \sum_{v \in V} \frac{1}{d(v)+1} \geq ??$$

CLAIM If $x_1, x_2 \dots x_k > 0$
then

$$\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_k} \geq \frac{k^2}{x_1 + \dots + x_k}$$

DO THIS NEXT

$$\frac{1}{\text{HARMONIC MEAN}} \geq \frac{1}{\text{ARITHMETIC MEAN}}$$

$x_i := d(v_i)+1$ $|V| = n$
$|E| = m$

$$\sum_{v \in V} \frac{1}{d(v)+1} \geq \frac{n^2}{\sum d(v) + n} = \frac{n^2}{2m+n}.$$

$$\boxed{\text{HARMONIC MEAN} \leq \text{GEOMETRIC MEAN} \leq \text{ARITH. MEAN}}$$