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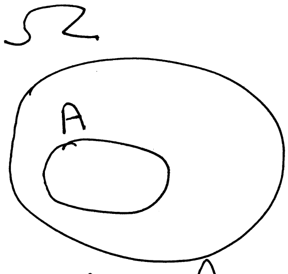
Conditional Expectation



$A \subseteq \Omega$

Z is a random variable = function

Average over A

$$E(Z \mid A) = \sum_{\omega \in A} Z(\omega)\, P(\omega \mid A)$$

"given" or "conditional on"

$$= \sum_{k} k \Pr(Z = k \mid A)$$

<u>Ex</u>

$S = X_1 + X_2 \qquad A = \{X_1 \geq X_2 + 4\}$

$A = \{(5,1), (6,1), (6,2)\} \qquad P(A) = \frac{1}{12}$

$E(Z \mid A) = 6 \times \frac{1}{3} + \ldots \qquad = 7$

$P((5,1) \mid A)$

"Average of averages = average"

$B_1, B_2, \ldots, B_n$ partition Ω

"Sample" events

$$E(Z) = \sum_{i=1}^{n} E(Z \mid B_i)\, P(B_i).$$

$$\sum_{i=1}^{n} E(Z \mid B_i)\, P(B_i) = \sum_{i=1}^{n} \sum_{\omega \in B_i} Z(\omega) \cdot \underbrace{\frac{P(\omega)}{P(B_i)}}_{P(\omega \mid B_i)} \times P(B_i)$$

$$= \sum_{i=1}^{n} \sum_{\omega \in B_i} Z(\omega)\, P(\omega)$$

partition

$$= \sum_{\omega \in \Omega} Z(\omega)\, P(\omega)$$

$$= E(Z)$$

First Moment Method

X is a random variable

$X : \Omega \to \{0, 1, 2, \ldots\}$

$$\Pr(X \geq 1) \leq E(X).$$

Proof

$B_1 = \{\omega \mid X(\omega) = 0\}$

B_2

$$E(X) = E(X \mid X = 0)\Pr(X = 0) + \underbrace{E(X \mid X \geq 1)}_{\geq 1}\Pr(X \geq 1)$$

$$\geq \Pr(X \geq 1)$$

Union Distinct Families

$\mathcal{A}$ is a family of subsets of $[n]$ $(\mathcal{A} \subseteq \mathcal{P}([n]))$ — power set

It is in union-free if for distinct $A, B, C, D \in \mathcal{A}$

$$A \cup B \neq C \cup D.$$

Suppose $\mathcal{A}$ consists of p randomly chosen subsets,

$X_1, X_2, \ldots, X_p$.

$Z = \#$ of i, j, k, l such that $X_i \cup X_j = X_k \cup X_l$

$\mathcal{A}$ is unionfree $\Leftrightarrow Z = 0$.

$P(Z \geq 1) < 1$?

$$E(Z) = p(p-1)(p-2)(p-3)\Pr(X_i \cup X_j = X_k \cup X_l),$$

$$< p^4 \left(\frac{5}{8}\right)^n$$

Choose a random set by flipping n coins.

$\frac{5}{8} = 1 - \frac{3}{8}$

$\frac{3}{8} = \frac{3}{16} + \underset{\sim}{\frac{3}{16}}$

coin $x = 1, 2, \ldots, n$

"mess" thing is if

$x \in (X_i \cup X_j) \setminus (X_k \cup X_l) = \frac{3}{4} \times \frac{1}{4}$

$P(x \notin X_i \cup X_j) = \frac{1}{4}$

$$E(Z) < p^4 \left(\frac{5}{8}\right)^n$$

So if $p \leqslant \left(\frac{8}{5}\right)^{n/4}$ then $E(Z) < 1$.

Minor Question

What if $\exists$ i, j : $A_i = A_j$

Then I get fewer than p sets.

$$\Pr(\exists i, j : A_i = A_j) \leqslant E(\underbrace{\# i < j}_{\binom{p}{2}} : A_i = A_j)$$

$$= \binom{p}{2} \frac{1}{2^n}$$

Choose p so that

$$p^4 \left(\frac{5}{8}\right)^n < \frac{1}{2} \quad \& \quad \binom{p}{2}\frac{1}{2^n} < \frac{1}{2}$$

Sort $x_1, x_2, \dots, x_n$

Choose p (randomly) [Other choices]

$L: x_i : x_i \leq x_p \qquad x_p \qquad R: x_i : x_i > x_p$

Apply Quicksort to and

How many comparisons:

T_n is a random variable. [$T_n \leq n^2$ worst-case]

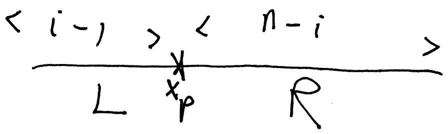
$E(T_n) \leq ?$

$$T_n \quad \sum_{i=1}^{n} E(\#\text{ comparisons} \mid p \text{ is the } i^{th} \text{ largest}) \times P(p \text{ is the } i^{th} \text{ largest})$$

Expectation

$$= \sum_{i=1}^{n} \left(T_{i-1} + T_{n-i} + \underbrace{n-1}\right) \times \frac{1}{n}$$

from producing L & R



Finding Minimum

$X = \#$ executions

$= X_1 + X_2 + \dots + X_n$

$X_i = \begin{cases} 1 & \text{if } i\text{th step requires execution} \\ 0 & \text{otherwise} \end{cases}$

$E(X) = E(X_1) + \dots + E(X_n)$

$= \Pr(X_1 = 1) + \dots + P(X_n = 1)$

$P(X_i = 1) = \frac{1}{i}$ $\quad X_i = 1 \Leftrightarrow$ x_i is smallest of $x_1, \dots, x_i$

$E(X) = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} = H_n$

$\approx \log_e n$