10/04/06

Random Variables

Any function $\xi : \Omega \to \mathbb{R}$

is a random variable.

Two Dice

$$\Omega = \{ (x_1, x_2) \}$$

$$\xi(x_1, x_2) = x_1 + x_2$$

$$P_k = \Pr(\xi = k) = P(\{ \omega : \xi(\omega) = k \})$$

Colored Indistinguishable Balls

$\Omega = \{ m \text{ balls}, n \text{ colours} \}$

$\xi(\omega) = \#$ of colors used

$\xi(RRWRBG) = 4$ $\qquad$ 3R, 1W, 1B, 1G

choose colors

$$p_k = \frac{\binom{n}{k}\binom{m-1}{k-1}}{|\Omega| = \binom{n+m-1}{n-1}}$$

$\# \; x_1 + \dots + x_k = m$
$x_i \geq 1$

Binomial $B(n,p)$

$p = Pr(H)$

$\Omega = (H,T)^n$

$B(n,p) = \#$ of heads

$B_{n,p}$

$$Pr(B(n,p) = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

k place for H

#choices for H

p_k

Poisson $Po(\lambda)$

$\Omega = \{0, 1, 2, \ldots\}$

$$Pr(Po(\lambda) = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

$B(n, \frac{\lambda}{n})$

$p = \frac{\lambda}{n}$

$$Pr(B(n, \tfrac{\lambda}{n}) = k)$$

Assume n is "large"
So H is rare.

$$= \binom{n}{k}\left(\frac{\lambda}{n}\right)^k\left(1 - \frac{\lambda}{n}\right)^{n-k}$$

$$\frac{n(n-1)\cdots(n-k+1)}{k!} \sim \frac{n^k}{k!}$$

$$\downarrow \qquad \frac{\cancel{n^k}}{k!} \cdot \frac{\lambda^k}{\cancel{n^k}} \cdot e^{(-\frac{\lambda}{n} \cdots)(n-k)}$$

$$\downarrow$$

$$e^{-\lambda}$$

Expectation

$$E(\xi) = \sum_{\omega \in \Omega} \xi(\omega) P(\omega)$$

$$= \sum_{k} k\, p_k$$

"Long term average value"

m balls n colors

$$E(\xi) = \sum_{k=1}^{n} \frac{k \binom{n}{k} \binom{m-1}{k-1}}{\binom{n+m-1}{n-1}}$$

Vandermonde Identity

$\vdots$

$$\frac{mn}{n+m-1}.$$

Geometric

$\Omega = \{1, 2, \ldots \}$

ζ = # coin flips until first H.

$$p_k = (1-p)^{k-1} p$$

$$E(\zeta) = \sum_{k=1}^{\infty} k(1-p)^{k-1} p$$

$$= p \sum_{k=1}^{\infty} k(1-p)^{k-1}$$

$$= \frac{p}{(1-(1-p))^2}$$

$$= \frac{1}{p}$$

Binomial $B_{n,p}$

$$E(B_{n,p}) = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

$$\vdots$$

$$np$$

$\underline{Po(\lambda)}$

$$E(Po(\lambda)) = \sum_{k=0}^{\infty} k \frac{\lambda^k e^{-\lambda}}{k!}$$

$$= \lambda \underbrace{\sum_{k=1}^{\infty} \frac{\lambda^{k-1} e^{-\lambda}}{(k-1)!}}_{1}$$

$$= \lambda$$

$B_{n,\frac{\lambda}{n}}$ $n \to \infty$

Suppose X & Y are random variables on Ω

$$E(X+Y) = E(X) + E(Y).$$

$$E(X_1+X_2+\cdots+X_n) = E(X_1) + E(X_2) + \cdots + E(X_n).$$

<u>Binomial</u>

$B_{n,p} = X_1 + X_2 + \cdots + X_n$ where $X_j = \begin{cases} 1 & j^{th} \text{ toss is } H \\ 0 & \text{otherwise} \end{cases}$

Indicator Variables

$$E(B_{n,p}) = E(X_1 + \cdots + X_n)$$
$$= E(X_1) + \cdots + E(X_n)$$

$$E(X_1) = 0 \times (1-p) + 1 \times p$$
$$= p$$

$$) = np$$

m balls n colors

$$Z_i = \begin{cases} 1 & \text{color } i \text{ is used} \\ 0 & \text{otherwise} \end{cases}$$

$Z = Z_1 + \dots + Z_n$ = # colors used

$$E(Z) = E(Z_1) + \dots + E(Z_n)$$
$$= n\,E(Z_1)$$
$$= n\,Pr(Z_1 \neq 0)$$
$$= n(1 - Pr(Z_1 = 0))$$
$$= n\left(1 - \frac{\binom{n+m-2}{m}}{\binom{n+m-1}{m}}\right)$$

$\leftarrow$ Total # colorings

$$= n\left(1 - \frac{n-1}{n+m-1}\right)$$

$$\frac{(n+m-2)!}{m!\,(n-2)!} \times \frac{m!\,(n-1)!}{(n+m-1)!}$$

$$= \frac{mn}{m+n-1}$$

Occurrence of pattern

Count # of times HTH occurs in $(H,T)^n$

HHTHHTT HTHTH

$S=3$

$$S = X_1 + X_2 + \cdots + X_{n-2}$$

$$X_i = \begin{cases} 1 & \text{HTH } i\text{th position} \\ 0 & \text{otherwise} \end{cases}$$

$$E(S) = E(X_1) + \cdots + E(X_{n-2})$$

$$= (n-2)\, p^2 (1-p)$$