09/27/06

Boole's Inequality

(Union Bound)

$$P(A \cup B) = P(A) + P(B) - \underbrace{P(A \cap B)}_{\geq 0}$$

$$\leq P(A) + P(B)$$

If $A \cap B = \emptyset$ [Disjoint events]

$$P(A \cup B) = P(A) + P(B)$$

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq P\left(\bigcup_{i=1}^{n-1} A_i\right) + P(A_n)$$

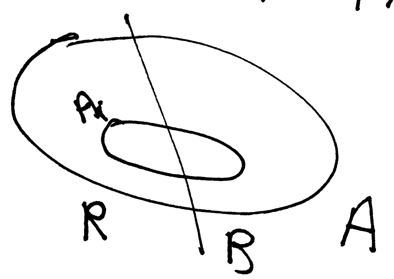
| induction

$$\leq P(A_1) + \cdots + P(A_{n-1}) + P(A_n)$$

Coloring Problem

$A_1, A_2, \ldots, A_n$ are subsets of A

Question: Does $\exists$ partition of $A = R \cup B$ such that $A_i \cap R \neq \emptyset$, $A_i \cap B \neq \emptyset$, $1 \leq i \leq n$



$A = \{1,2,3,4\}$ $\qquad$ $A_1 = \{1,2,3\}$ $\qquad$ $B_1 = \{2,3,4\}$

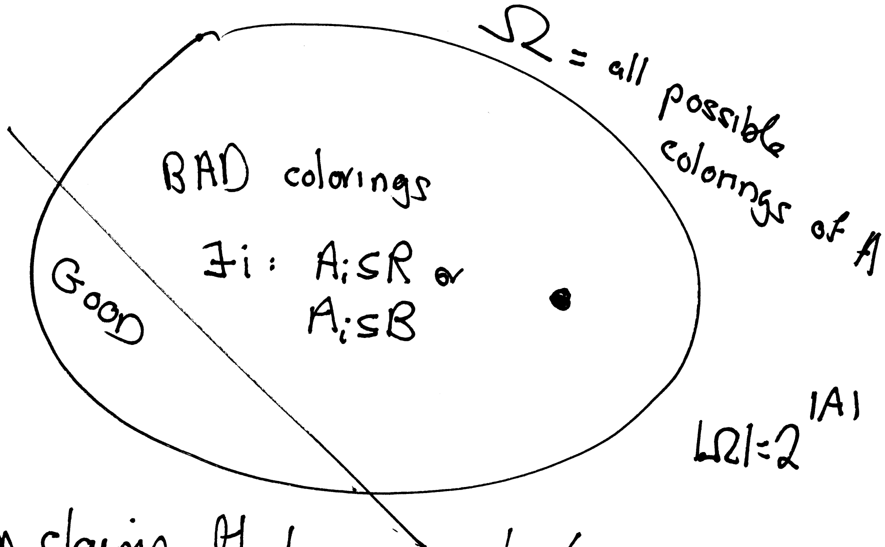
$\qquad\qquad\qquad\qquad\qquad$ R R B $\qquad\qquad$ R B B

Thm

If $|A_i| = k \ \forall i$ and $n < 2^{k-1}$

then a colouring exists.

PROBABILISTIC METHOD $\qquad$ PAUL ERDŐS



Theorem claims that $\text{GOOD} \neq \emptyset$

If you make Ω into a probability space and make each colouring equally likely then

$$P(\text{GOOD}) > 0$$

or

$$P(\text{BAD}) < 1$$

$$P(\text{GOOD}) + P(\text{BAD}) = 1$$

$$P(BAD) < 1$$

$$BAD = \bigcup_{i=1}^{n} BAD_i$$

$$\{A_i \subseteq R \text{ OR } A_i \subseteq B\}$$

$$P(BAD) \leqslant \sum_{i=1}^{n} P(BAD_i)$$

To choose an element from Ω, I randomly color each element of A.

$$P(BAD) \leqslant \sum_{i=1}^{n} \left(\frac{1}{2^k} + \frac{1}{2^k} \right) = \frac{n}{2^{k-1}} < 1$$

$\square$

Can you always 2-colour $A_1, \dots, A_n$?

$A = [2k-1]$ $\quad n = \binom{2k-1}{k} > 2^{k-1}$.

$A_1, A_2, \dots, A_n = \binom{A}{k}$

In any 2-coloring of A, there is mono-chromatic set.

$$A = R \cup B$$

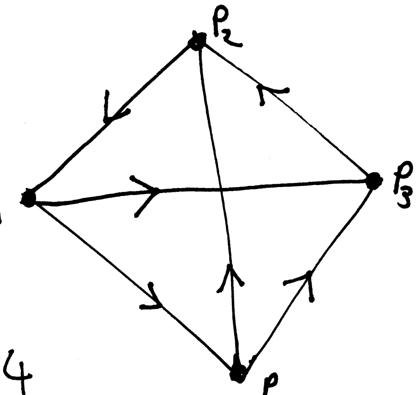
$$|R| \geq k \quad \text{or} \quad |B| \geq k$$

Assume $|R| \geq k$ then

any k-subset of R is all Red.

Tournaments

n players. Each play each other. No ties.



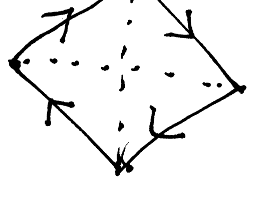
"Orientation of K_n"

$n=4$

K_4

Fix k. Does $\exists$ a tournament such if $|S|=k$ then $\exists$ player w_S who beats all of them.

$k=1$: easy



✓

$k=2$: think about it.

We prove the existence of a tournament by showing that a random tournament has the property with positive probability.

BAD: $\exists S$ s.t. nobody beat S

$$BAD = \bigcup_{\substack{S \subseteq [n] \\ |S|=k}} BAD_S$$

$$Pr(BAD) \leq \sum_{\substack{S \subseteq [n] \\ |S|=k}} P(BAD_S)$$

$$\left(1-\frac{1}{2^k}\right)^{n-k}$$

S

$Pr(x \text{ beats } S) = \frac{1}{2^k}$

$Pr(x \text{ does not beats } S) = 1-\frac{1}{2^k}$

$$P(\text{BAD}) \leqslant \binom{n}{k}\left(1-\frac{1}{2^k}\right)^{n-k}$$

k is fixed (say 100), n can be arbitrarily large

$$\leqslant n^k e^{-\frac{n-k}{2^k}}$$

$1-x \leqslant e^{-x}$

$$\longrightarrow 0$$

$$k \log n \ll \frac{n-k}{2^k}$$