09/25

Discrete Probability

Ω is finite or countable set.

$P : \Omega \to \mathbb{R}^+$

$$\sum_{\omega \in \Omega} P(\omega) = 1$$

Geometric Distribution

$\Omega = \{1, 2, 3, \ldots \}$

$0 \leq p \leq 1 = Pr[\text{Success}]$

$P[k] = (1-p)^{k-1} p$

Formally

$\Omega = \{S, FS, FFS, FFFS, \ldots \}$

An <u>event</u> is a subset of Ω.

I pick I, $|I| = 7$

PA chooses J, $|J| = 11$.
Randomly

$$\Omega = \binom{[80]}{11}$$

$$\text{WIN} = \{ I \subseteq J \}$$

$$\Pr[\text{WIN}] = \frac{|\text{WIN}|}{|\Omega|} = \frac{\binom{73}{4}}{\binom{80}{11}}$$

$|\text{WIN}|$ = all sets of size 11, that contain I.

n days in a year

$\Omega = [n]^k$

$\omega = b_1 \, b_2 \, b_3 \, \ldots \, b_k$

Distinct $= \{ b_i \neq b_j, \text{ for all } i, j \}$

$$\Pr[D] = \frac{n(n-1)(n-2)\cdots(n-k+1)}{n^k = |\Omega|}$$

Balls in Boxes (Urn Models)

m distinguishable balls,
place them randomly into n
boxes.

$E = \{\text{Box } 1 \text{ is empty}\}$

$$Pr[E] = \frac{|E| = (n-1)^m}{|\Omega| = n^m} = \left(1 - \frac{1}{n}\right)^m$$

$\rightarrow e^{-c}$

if $m = cn$

$b_1 \; b_2 \; b_3 \ldots \; b_m$

$\uparrow \; \uparrow \; \uparrow$

choose
number
between
1 and n

$n \times n \times \ldots \times n$

Very Useful Inequality

$$1 + x \leq e^x \qquad \forall x$$

1(a) $x \geq 0$

$$1+x \leq \underbrace{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots}_{e^x}$$

(b) $x < -1$: $1+x < 0 \leq e^x$

$\uparrow \geq 0$

(c) $-1 \leq x < 0$: $\quad 1-y \qquad e^{-y}$

$\uparrow$ $-y$

$$1 - y + \underbrace{\left[\frac{y^2}{2} - \frac{y^3}{6}\right]}_{\geq 0} + \underbrace{\left[\frac{y^4}{24} - \frac{y^5}{120}\right]}_{\geq 0} + \cdots$$

Birthday Paradox

$$Pr[D] = \frac{n(n-1)\ldots(n-k+1)}{n^k} = \prod_{i=1}^{k-1}\left(1 - \frac{i}{n}\right) \leq \prod_{i=1}^{k-1} e^{-i/n}$$

$$= \exp\left\{-\sum_{i=1}^{k-1} \frac{i}{n}\right\}$$

$$= \exp\left\{-\frac{k(k-1)}{2n}\right\}$$

If $k \gg \sqrt{2n}$ then $\Pr[D]$ is small.

So if $k(k-1) \geq 4n$ ie $k \geq 2\sqrt{n}$

$\Pr[D] \leq e^{-2}$