9\15\06

$$a(x) = \frac{-1/2}{1-x} - \frac{1}{(1-x)^3} + \frac{5/2}{1-3x}$$

$$= -\frac{1}{2}\sum_{n=0}^{\infty} x^n - \sum_{n=0}^{\infty} \binom{n+2}{2} x^n + \frac{5}{2}\sum_{n=0}^{\infty} 3^n x^n$$

$$\left[\frac{1}{(1-x)^m} = \sum_{n=0}^{\infty} \binom{m+n-1}{n} x^n\right]$$

$$a_n = -\frac{1}{2} - \binom{n+2}{2} + \frac{5}{2} \cdot 3^n$$

$$\sum_{n=k}^{\infty}\left(a_n + c_1 a_{n-1} + \cdots + c_k a_{n-k}\right) x^n = \underbrace{\sum_{n=k}^{\infty} u_n x^n}_{U(x)}$$

$$a(x) = \frac{U(x) + r(x)}{q(x)}$$

$$q(x) = 1 + c_1 x + \cdots + c_k x^k = \prod_{i=1}^{k} (1 - \alpha_i x)$$

Product of generating functions

$$a(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$b(x) = \sum_{n=0}^{\infty} b_n x^n$$

$$a(x)\, b(x) = \sum_{n=0}^{\infty} c_n x^n$$

$$c_n = ?$$

$$(a_0 + a_1 x + a_2 x^2 + \dots)(b_0 + b_1 x + b_2 x^2 + \dots)$$

$$c_0 = a_0 b_0$$

$$c_1 = a_0 b_1 + a_1 b_0$$

$$c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0$$

$$\vdots$$

$$c_n = a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k}$$

$$\frac{1}{(1-x)^2} = \frac{1}{1-x} \times \frac{1}{1-x}$$

$$a(x) \qquad b(x)$$

$$a_n = 1 \qquad b_n = 1$$

$$c_n = \sum_{k=0}^{n} 1 \times 1 = n+1$$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1) x^n$$

$$\frac{1}{(1-x)^3} = \frac{1}{(1-x)^2} \times \frac{1}{1-x}$$

$$a_n = n+1 \qquad b_n = 1$$

$$c_n = \sum_{k=0}^{n} (k+1) \times 1 = (n+1)(n+2)/2$$

$$= \binom{n+2}{2}$$

Derangements

d_n = # of derangements.

$$n! = \sum_{k=0}^{n} \underbrace{\binom{n}{k} d_{n-k}}$$

# of permutations of $[n]$

$$|S_k| = \binom{n}{k} d_{n-k}$$

S_k = { permutations where $\pi(i) = i$, k time }

$k=0$ for derangement

$\underline{n=10}$

i	1	2	3	4	5	6	7	8	9	10
$\pi(i)$	2	1	3*	5	4	6*	10	8*	7	9

$k=3$

$$n! = \sum_{k=0}^{n} \binom{n}{k} d_{n-k}$$

$$= \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} d_{n-k}$$

$$1 = \sum_{k=0}^{n} \frac{1}{k!} \times \frac{d_{n-k}}{(n-k)!} \qquad n \geq 0$$

General method: mulipy by x^n and add

$$\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \Big(\underbrace{\frac{1}{k!}}_{a_k} \times \underbrace{\frac{d_{n-k}}{(n-k)!}}_{b_{n-k}} \Big) x^n$$

$$\frac{1}{1-x}$$

$a_k = \frac{1}{k!}$

$b_{n-k} = \frac{d_{n-k}}{(n-k)!}$

Product!

$$\underbrace{\sum_{n=0}^{\infty} \underbrace{\frac{x^n}{n!}}_{a_n}}_{e^x} \times \underbrace{\sum_{n=0}^{\infty} \underbrace{\frac{d_n}{n!}}_{b_n} x^n}_{d(x)}$$

$$\frac{1}{1-x} = e^{x}\, d(x)$$

$$d(x) = \frac{\overset{a(x)}{e^{-x}}}{\underset{b(x)}{1-x}}$$

product

$$\frac{d_n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \times 1$$

$$d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$