09/13/06

$$\underbrace{a(x)-1-9x}_{\text{first sum}} - \underbrace{6x(a(x)-1)}_{2^{nd}\text{ sum}} + 9x^2 a(x) = 0$$

$\vdots$

$$a(x) = \frac{1+3x}{(1-3x)^2}$$

$$= a_0 + a_1 x + a_2 x^2 + \ldots$$

$$= \frac{1}{(1-3x)^2} + \frac{3x}{(1-3x)^2}$$

$$= \sum_{n=0}^{\infty} (n+1)(3x)^n + \sum_{n=0}^{\infty} (n+1)(3x)^{n+1}$$

a_n

$$= \sum_{n=0}^{\infty} \left((n+1)3^n + n3^n\right) x^n$$

$$\frac{1}{(1-y)^2} = 1 + 2y + 3y^2 + \cdots + (n+1)y^n + \cdots$$

$(2n+1)3^n$

$$\sum_{n=2}^{\infty}(a_n - a_{n-1} - a_{n-2})x^n = 0$$

$$\Downarrow$$

$$\sum_{n=2}^{\infty} a_n x^n - x\sum_{n=2}^{\infty} a_{n-1}x^{n-1} - x^2\sum_{n=2}^{\infty} a_{n-2}x^{n-2} = 0$$

$$\uparrow$$

$$a(x) - 1 - x \quad - x\left(a(x) - 1\right) \quad - x^2 a(x)$$

$$\Downarrow$$

$$a(x) = \frac{1}{1-x-x^2}$$

$$= \frac{1}{-(\xi_1 - x)(\xi_2 - x)}$$

where ξ_1, ξ_2 are the roots of

$$x^2 - x - 1 = 0$$

$$a(x) = \frac{1}{-(\xi_1 - x)(\xi_2 - x)}$$

$$= \frac{1}{\xi_1 - \xi_2}\left[\frac{1}{\xi_1 - x} - \frac{1}{\xi_2 - x}\right]$$

Where from?

Write

$$\frac{-1}{(\xi_1 - x)(\xi_2 - x)} = \frac{A}{\xi_1 - x} + \frac{B}{\xi_2 - x} \quad \forall x$$

$$-1 = A(\xi_2 - x) + B(\xi_1 - x), \ \forall x$$

$$= -(A+B)\,x + A\xi_2 + B\xi_1, \quad \forall x$$

$$A + B = 0$$
$$A\xi_2 + B\xi_1 = 1$$

Solve for A, B

Partial Fractions

$$a(X) = \frac{1}{\xi_1 - \xi_2}\left[\frac{1}{\xi_1 - X} - \frac{1}{\xi_2 - X}\right]$$

$$= \frac{1}{\xi_1 - \xi_2}\left[\frac{1/\xi_1}{1 - X/\xi_1} - \frac{1/\xi_2}{1 - X/\xi_2}\right]$$

$$= \frac{1}{\xi_1 - \xi_2}\left[\frac{1}{\xi_1}\sum_{n=0}^{\infty}\frac{X^n}{\xi_1^n} - \frac{1}{\xi_2}\sum_{n=0}^{\infty}\frac{X^n}{\xi_2^n}\right]$$

$$a_n = [x^n](\quad)$$

$$= \frac{1}{\xi_1 - \xi_2}\left[\frac{1}{\xi_1^{n+1}} - \frac{1}{\xi_2^{n+1}}\right]$$

$$\xi_1 = -\frac{\sqrt{5}+1}{2}$$

$$\xi_2 = \frac{\sqrt{5}-1}{2}$$

$$\xi_1 - \xi_2 = -\sqrt{5},$$

$$= -\frac{1}{\sqrt{5}}\left(\left(\frac{-2}{\sqrt{5}+1}\right)^{n+1} - \left(\frac{2}{\sqrt{5}-1}\right)^{n+1}\right)$$

$$\frac{1}{\sqrt{5}+1} = \frac{\sqrt{5}-1}{(\sqrt{5}+1)(\sqrt{5}-1)} = \frac{\sqrt{5}-1}{4} \quad \cdots\cdots$$

$$a_n - 3a_{n-1} = n^2 \qquad n \geq 1$$

$a_0 = 1$

$$\sum_{n=1}^{\infty} (a_n - 3a_{n-1}) x^n = \sum_{n=1}^{\infty} n^2 x^n$$

$$\sum_{n=1}^{\infty} a_n x^n - 3\sum_{n=1}^{\infty} a_{n-1} x^n = \sum_{n=1}^{\infty} n^2 x^n$$

$$\downarrow$$

$$a(x) - 1 \; - 3x\,a(x) = \sum_{n=1}^{\infty} n(n-1)x^n + \sum_{n=1}^{\infty} n x^n$$

$$= \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2}$$

$$\frac{1}{1-y} = 1 + y + y^2 + y^3 + \cdots \qquad y^n$$

$$\frac{1}{(1-y)^2} = 1 + 2y + 3y^2 + \cdots \qquad n y^{n-1}$$

$$\frac{2}{(1-y)^3} = 2 + \qquad n(n-1) y^{n-2}$$

$$a(x) = \frac{x+x^2}{(1-x)^3(1-3x)} + \frac{1}{1-3x}$$

$$= \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{(1-x)^3} + \frac{D}{1-3x}$$

Find A, B, C, D

$$(x+x^2) \qquad + (1-x)^3$$

$$= A(1-x)^2(1-3x) + B(1-x)(1-3x) + C(1-3x) + D(1-x)^3$$

Put $x=1$

$$2 = -2C \Rightarrow C = -1$$

Put $x = \frac{1}{3}$

$$\frac{20}{27} = D\frac{8}{27} \Rightarrow D = \frac{5}{2} \qquad (D+1 \text{ in slide})$$