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Derangements

Permutations $\pi : [n] \to [n]$

$$\pi(i) \neq i \quad , \forall i$$

i:	1	2	3	4
$\pi(i)$:	2	3	4	1

Derangement.

$D_n = \#$ $= ?$

D_3

$$\begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{matrix} \quad \text{or} \quad \begin{matrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{matrix}$$

$D_3 = 2$

$A = \{\text{permutations}\}$

$|A| = n!$

π is a derangement if

$(\text{not } 1) \wedge (\text{not } 2) \wedge \ldots$

property 1 property 2

To apply IE

express target set [Der...]

as elements which

do <u>not</u> have property 1

and <u>not</u> " " 2

:

For derangements

Property i is "$\pi(i)=i$"

$$A_i = \{\pi : \pi(i) = i\}$$

$$D_n = \left|\bigcap_{i=1}^{n} \overline{A_i}\right|$$

$$= \sum_{S \subseteq [n]} (-1)^{|S|} |A_S|.$$

$$|A_\emptyset| = |A| = n!$$

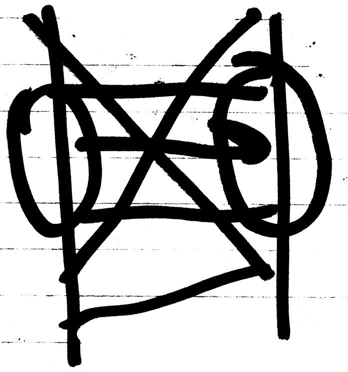
$$|A_{\{1\}}| = |\{\pi : \pi(1) = 1\}|$$

$$= (n-1)!$$

$$|A_{\{1,2\}}| = \left\{\pi : \begin{array}{l} \pi(1)=1 \\ \pi(2)=2 \end{array}\right\}$$

$$= (n-2)!$$

$$|A_S| = |\{\pi : \pi(i)=i,\ i \in S\}|$$

S  $=(n-|S|)!$

$$D_n = \sum_{S\subseteq[n]} (-1)^{|S|}(n-|S|)!$$

$$= \sum_{k=0}^{n} (-1)^k \binom{n}{k}(n-k)!$$

$\#$ of k-sets

$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \cdot (n-k)!$$

$$= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

$$= n! \underbrace{\left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots \frac{(-1)^n}{n!}\right)}_{\text{Proportion of } \pi \text{ that are derangements}}$$

$\downarrow e^{-1}$ as $n \to \infty$.

Exercise:

Prove $\underline{IE}$ formula by induction on N.

Algebraic Proof

$$\theta_{x,i} = \begin{cases} 1 & x \in A_i \\ 0 & x \notin A_i \end{cases}$$

$$(1-\theta_{x,1})(1-\theta_{x,2})\cdots(1-\theta_{x,N})$$

$$= \begin{cases} 1 : x \in \bigcap_i \bar{A}_i \\ 0 : \text{otherwise} \end{cases}$$

$$\left|\bigcap_{i=1}^{N} \bar{A}_i\right| = \sum_{x\in A} (1-\theta_{x,1})\cdots(1-\theta_{x,N})$$

$$= \sum_{x\in A}\sum_{S\subseteq[N]} (-1)^{|S|}\prod_{i\in S}\theta_{x,i}$$

$$= \sum_{S\subseteq[N]} (-1)^{|S|} \sum_{x\in A}\underbrace{\prod_{i\in S}\theta_{x,i}}_{=\begin{cases}1 & x\in A_S\\ 0 & \text{otherwise}\end{cases}}$$

$$\sum_{x\in A}\prod_{i\in S}\theta_{x,i} = |A_S|$$

$$A = \{ f : [n] \to [m] \}$$

(I) $|A| = m^n$

$\left\{ \begin{array}{l} \text{We did 1-1 functions} \\ \quad m(m-1)\ldots(m-n+1) \end{array} \right.$

(II) Onto functions

(Surjection)

$A = \{\text{all functions}\}$

$$
\begin{aligned}
\{\text{surjections}\} &= \bigcap \overline{A}_i \\
&= \{ f : \text{ not in } A_1 \leftarrow \{ f : f \text{ misses } 1 \} \\
&\qquad \text{and not in } A_2 \\
&\qquad \text{and not in } A_3 \\
&\qquad \vdots \\
&\quad \}
\end{aligned}
$$

$$A_i = \{ f : f(x) \neq i, \forall x \}$$

$$F(n,m) = \left| \bigcap_{i=1}^{m} \overline{A}_i \right|$$

$$= \sum_{S \subseteq [m]} (-1)^{|S|} |A_S|$$

$$\{ f : f(x) \notin S \ \forall x \in [n] \}$$

$$= \{ f : [n] \to [m] \setminus S \}$$

$$|A_S| = (m-|S|)^n$$

$$F(m,n) = \sum_{S \subseteq [m]} (-1)^{|S|} (m-|S|)^n$$

$$= \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)^n$$

Related:

Stirling No's 2^{nd} Kind