# 21-301 Combinatorics
# Homework 9
# Due: Monday, November 14

1. Prove that if we 2-color the edges of $K_n$ then either (i) there is a vertex of Red degree at least $\frac{n}{2}-1$ or (ii) there is a Blue triangle. Show also that it is posible to have a 2-coloring in which the maximum Red degree is $\frac{n}{2}-1$ and in which there is no Blue triangle.

   **Solution** If there is no vertex of Red degree $\geq \frac{n}{2}-1$ then every vertex has minimum Blue degree $\geq \frac{n}{2}+1$. Thus the number of Blue edges is greater than $\frac{n^2}{4}$ and so by Turan's Theorem (Graph Theory Notes p60) there is a Blue triangle.

   **Alternative proof** Let $(u, v)$ be a Blue edge. Both $u, v$ have at least $\frac{n}{2}$ Blue neighbors outside $u, v$. This means they have a common Blue neighbor.

   If $n = 2m$ we can split the vertex set $[n]$ into two sets $A, B$ of size $m$. Then if we color edges inside $A$ or inside $B$ Red and edges between $A, B$ Blue then *every* vertex has Red degree $\frac{n}{2}-1$ and there is no Blue triangle.

2. Prove that if we 2-color the edges of $K_6$ then there are least *two* monochromatic triangles.

   **Solution** Assume w.l.o.g. that triangle $(1, 2, 3)$ is Red and that $(4, 5, 6)$ is not Red and in particular that edge $(4, 5)$ is Blue. If $x = 4, 5$ or $6$ then there can be at most one Red edge joining $x$ to $1, 2, 3$, else we get a Red triangle. So we can assume that there are two Blue edges joining each of $4, 5$ to $1, 2, 3$. So there must be $x \in \{1, 2, 3\}$ such that both $(x, 4)$ and $(x, 5)$ are Blue. But then triangle $(x, 4, 5)$ is Blue.

   **Alternative proof** Given the coloring, let us count the number $N$ of paths of length two which consist of a Red edge followed by a Blue edge. Let $r_i$ denote the number of Red edges incident with $i$. Then we have

$$N = \sum_{i=1}^{6} r_i(6 - r_i) \leq \sum_{i=1}^{6} 6 = 36.$$

   Each of the 20 triangles of $K_6$ contains 0 or 2 of these paths. So at most 18 contain 2 and there are at least 2 mono-colored triangles.

3. Prove that if $n \geq R(2k, 2k)$ and if we 2-color the edges of $K_{n,n}$ then there is a mono-chromatic copy of $K_{k,k}$.

   **Solution** Given a coloring $\sigma$ of $K_{n,n}$ we construct a coloring $\tau$ of the edges of $K_n$ as follows. If $i < j$ then we give the edge $(i, j)$ of $K_n$ the same color that is given to edge $(i, j)$ under $\sigma$.

   Since $n \geq R(2k, 2k)$ we see that $K_n$ contains a mono-colored copy of $K_{2k}$. If the set of vertices of this copy is $S$, divide $S$ into two parts $S_1, S_2$ of size $k$ where $\max S_1 < \min S_2$. It follows that the bipartite sub-graph of $K_{n,n}$ defined by $S_1, S_2$ is mono-colored under $\sigma$.