21-301 Combinatorics
Homework 7
Due: Monday, November 8

1. Let $s \geq 1$ be fixed. Let $\mathcal{A} \subseteq \mathcal{P}_n$ be such that **there do not exist** distinct $A_1, A_2, \ldots, A_{s+1}$ such that $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_{s+1}$. Show that

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq s.$$

Solution Let π be a permutation of $[n]$ and for $A \in \mathcal{A}$ let

$$1_A = \begin{cases} 1 & if\ \{\pi(1), \pi(2), \ldots, \pi(|A|)\} = A \\ 0 & otherwise \end{cases}$$

Our condition ensures that for all π,

$$\sum_{A \in \mathcal{A}} 1_A \leq s.$$

Now let π be a random permutation. Then

$$s \geq \mathbf{E}\left(\sum_{A \in \mathcal{A}} 1_A\right) = \sum_{A \in \mathcal{A}} \mathbf{E}(1_A) = \frac{1}{\binom{n}{|A|}}.$$

2. Show that if $n \geq 1$ and $S \subseteq [2n]$ and $|S| \geq n+1$ then there exist distinct $a, b \in S$ such that a divides b.

Solution Positive integer i can be expressed $i = 2^k j$ for some $k \geq 0$ and *odd* j. Put i into box j and note that the numbers $1, 2, \ldots, 2n$ occupy the n boxes corresponding to $1, 3, \ldots, 2n-1$. By the pigeon-hole principle, S contains two numbers which go into the same box.

3. Let $m = n^4 + 1$. Given two sequences $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_m$ of real numbers, show that there exist $i_1, i_2, \ldots, i_{n+1}$ such that *both* subsequences $a_{i_1}, a_{i_2}, \ldots, a_{i_{n+1}}$ and $b_{i_1}, b_{i_2}, \ldots, b_{i_{n+1}}$ are monotone.

Here we do not insist that both are monotone increasing or both are monotone decreasing, so one can be increasing and the other decreasing.

Solution It follows from the Erdős-Szekeres Theorem that there exist $i_1, i_2, \ldots, i_{n^2+1}$ such that $a_{i_1}, a_{i_2}, \ldots, a_{i_{n^2+1}}$ is a monotone sequence. It follows from the Erdős-Szekeres Theorem that $b_{i_1}, b_{i_2}, \ldots, b_{i_{n^2+1}}$ contains a monotone subsequence of length $n+1$. If the corresponding indices are $j_1, j_2, \ldots, j_{n+1}$ then $a_{j_1}, a_{j_2}, \ldots, a_{j_{n+1}}$ and $b_{j_1}, b_{j_2}, \ldots, b_{j_{n+1}}$ are both monotone.