21-301 Combinatorics
Homework 7
Due: Monday, October 31

1. Show without using the Cayley formula that if T_n denotes the number of trees on n given vertices then

$$(n-1)T_n = \sum_{k=1}^{n-1} k(n-k)\binom{n-1}{k-1} T_k T_{n-k}.$$

Solution First observe that if we delete an edge from a tree then we get a forest with exactly 2 tree components. This is because we have an n-vertex acyclic graph with $n-2$ edges.

The term $(n-1)T_n$ is the number of ways of choosing a tree T and an edge $e \in E(T)$.

To obtain the RHS, we see that we can choose e, T by first choosing two vertex disjoint trees G_1, G_2 where $1 \in V(G_1)$ and $1 \leq k = |V(G_1)| \leq n-1$. Having chosen k, we can choose the vertices of G_1 in $\binom{n-1}{k-1}$ ways and then G_1, G_2 in $T_k T_{n-k}$ ways. We can then choose e and $T = T_1 + T_2 + e$ in $k(n-k)$ ways.

2. Show that there are $2^{\binom{n-1}{2}}$ graphs on vertex set $[n]$ which do not have any vertices of odd degree.

We count these graphs by counting their adjacency matrices A. We put 0 or 1 into $A(i,j)$ for $1 \leq i < j \leq n-1$ in $2^{\binom{n-1}{2}}$ ways. We must then choose $A(i,n)$ uniquely in order to make the degree of vertex i even. Finally, we observe that the degree of n is automatically even, because otherwise we would have defined a graph with a unique odd vertex, making the number of odd vertices odd.

3. Every pair of odd cycles of the graph G intersect in at least one vertex. Show that G has chromatic number at most 5.

Solution Let C be an odd cycle of G. If there are none then G is bipartite and can be 2-colored. Let H be the graph obtained by deleting the vertices of C from G. H has no odd cycles because every odd cycle of G loses at least one vertex in the deletion of G. Thus H is bipartite and can be 2-colored. C itself can be 3-colored and 2+3=5.