21-301 Combinatorics
Homework 6
Due: Monday, October 24

1. Let P_1, P_2 be two paths of maximum length in a connected graph G. Prove that P_1, P_2 share a common vertex.

   Solution Suppose that P_1, P_2 are vertex disjoint. Let $P_1 = (u_1, u_2, \ldots, u_k)$ and $P_2 = (v_1, v_2, \ldots, v_k)$. Let $Q = (u_i = w_1, w_2, \ldots, w_\ell = v_j)$ be a shortest path from a vetex in P_1 to a vertex in P_2. Such a path exists because G is connected. Now $w_2, w_3, \ldots, w_{\ell-1}$ are not vertices of P_1 or P_2, since Q is a shortest path. Assume w.l.o.g. that $i, j \geq k/2$. Then the path $u_1, u_2, \ldots, u_i, w_2, \ldots, w_{\ell-1}, u_j, u_{j-1}, \ldots, u_1$ has length at least $i + j + 1 \geq k + 1$, contradiction.

2. Show that if $G = (V, E)$ is not connected then its complement $\bar{G} = (V, \bar{E})$ is connected. ($\bar{E} = \{(x, y) : x, y \in V \text{ and } (x, y) \notin E\}$).

   Solution Let u and v be two vertices of G. If they are in different components of G, they will be adjacent in $\bar{G}$. If they are in the same component in G, they will both be adjacent to any vertex in any other component of G (such a component exists, since G is not connected). So, they will be joined by a path of length 2.

3. Let T be an arbitrary tree on $k+1$ vertices. Show that if $\delta(G) \geq k$ then G has a subgraph isomorphic to T.

   Solution Label any leaf of T, v_1. Map v_1 to any vertex h_1 in G. Now repeat the following until all vertices of T are mapped to vertices of G. Suppose we have labeled $i - 1$ vertices of T. Take any neighbor w of an already labeled vertex in T, and label it v_i. There is exactly 1 v_j adjacent to v_i, with $j < i$ because there are no cycles in T. (The set of already labelled vertices forms a connected sub-graph of T). Now map v_i to any *new* vertex adjacent to h_j. (Such a vertex will always be available because we couldn't have used more than $i - 1 < k$ neighbors of any vertex in G.) When we have mapped all vertices of T into G, the subgraph of G defined on these vertices will be isomorphic to T.