21-301 Combinatorics
Homework 5
Due: Monday, October 10

1. A box has three drawers; one contains two gold coins, one contains two silver coins and one contains one gold and one silver coin. Assume that one drawer is selected randomly and that a randomly selected coin from that drawer turns out to be gold. What is the probability that the chosen drawer is the one with two gold coins?

   Solution Let the three drawers be A, B, C. Let G, S stand for the chosen coin being Gold/Silver respectively. Then what we want is

   $$\Pr(A \mid G) = \frac{\Pr(A \wedge G)}{\Pr(G)}.$$

   Now

   $$\Pr(A \wedge G) = \Pr(A) = \frac{1}{3}.$$

   $$\begin{aligned}\Pr(G) &= \Pr(G \mid A)\Pr(A) + \Pr(G \mid B)\Pr(B) + \Pr(G \mid C)\Pr(C) \\ &= 1 \times \frac{1}{3} + \frac{1}{2} \times \frac{1}{3} + 0 \times \frac{1}{3} \\ &= \frac{1}{2}.\end{aligned}$$

   So

   $$\Pr(A \mid G) = \frac{1/3}{1/2} = \frac{2}{3}.$$

2. A bag contains n balls, each of a different color. In a round, a person picks a random ball from the bag, makes a note of its color and then puts it back. What is the expected number of rounds required for the person to have pulled out a ball of each color at least once?

   Solution Let T_i be the number of rounds needed to increase the number of different colors chosen so far from $i-1$ to i. Thus $T_1 = 1$ but in general T_i is a random variable and the question asks for $\mathbf{E}(T_1 + T_2 + \cdots + T_n)$.

   Now when $i - 1$ colors have been chosen, the probability that we see a new one on the next round is $\frac{n-i+1}{n}$, regardless of the previous drawings.

Thus T_i is distributed as a geometric random variable with probability of success $\frac{n-i+1}{n}$. Thus

$$\mathbf{E}(T_i) = \frac{n}{n-i+1}$$

and the expected total number of drawings is

$$n\sum_{i=1}^{n} \frac{1}{n-i+1} = n\sum_{i=1}^{n} \frac{1}{i}.$$

3. A particle sits at the left hand end of a line $0 - 1 - 2 - \cdots - L$. When at 0 it moves to 1. When at $i \in [1, L-1]$ it makes a random move to $i-1$ or $i+1$ with equal probability. When at L it stops. Show that the expected number of visits to 0 during the whole random walk is L. Here the count of visits starts at 1 with the particle being at 0.

   (Hint: Let E_k denote the expected number of visits if we started the walk at k. Construct a set of equations satisfied by the E_k's. What we want is E_0.)

   Solution The equations are

$$\begin{aligned} E_L &= 0 \\ E_0 &= 1 + E_1 \\ E_k &= \frac{E_{k-1} + E_{k+1}}{2} \end{aligned}$$

   for $0 < k < L$.

   These equations have the *unique* solution $E_k = L - k$.