21-301 Combinatorics
Homework 4
Due: Monday, October 3

1. $3n$ distinguishable balls are independently randomly numbered with 1,2 or 3, each number being equally likely. What is the probability that there are n balls with number 1, and n balls with number 2 and n balls with number 3.

   Solution The number of ways of choosing $n+n+n$ balls of each color is $\binom{3n}{n,n,n}$. The number of ways of colouring the balls is 3^{3n}. Thus the probability is
$$\frac{\binom{3n}{n,n,n}}{3^{3n}}.$$

2. Let $A_1, A_2, \ldots, A_m$ be subsets of A and $|A_i| = n$. Show that if $m < \frac{4^{n-1}}{3^n}$ then there is a way of coloring A with 4 colors so that each color appears at least once in each set.

   Solution Let $\mathcal{E}_{i,j}$ be the event that color j is not used on A_i and let $\mathcal{E}_i = \bigcup_{j-1}^{4} \mathcal{E}_{i,j}$ denote the event that E_i is colored with fewer than 4 colors. Then
$$\Pr(\mathcal{E}_i) \leq \sum_{j=1}^{4} \Pr(\mathcal{E}_{i,j}) = 4\left(\frac{3}{4}\right)^n.$$

   Thus,
$$\Pr\left(\bigcup_{i=1}^{m} \Pr(\mathcal{E}_i)\right) \leq 4m\left(\frac{3}{4}\right)^n < 1$$
   and so there is a coloring for which none of the $\mathcal{E}_i$ occur.

3. Let $s_1, s_2, \ldots, s_m$ be binary strings such that no string is a prefix of another string.
   ($a = a_1a_2\cdots a_p$ is a prefix of $b = b_1b_2\cdots b_q$ if $p \leq q$ and $a_i = b_i$ for $1 \leq i \leq p$).

   Show that
$$\sum_{i=1}^{m} 2^{-|s_i|} \leq 1$$

where $|s|$ is the length of string s.

(Hint: Let $n = \max\{|s_1| : 1 \leq i \leq n\}$. Let x be a random binary string of length n. Consider the events $\mathcal{E}_i = \{s_i$ is a prefix of x.)

Solution The events $\mathcal{E}_i$ of the hint are *disjoint*. This follows from the assumption that no string is a prefix of another. Thus

$$1 \geq \sum_{i=1}^{m} \Pr(\mathcal{E}_i) = \sum_{i=1}^{m} 2^{-|s_i|}.$$