21-301 Combinatorics
Homework 4
Due: Monday, October 3

1. $3n$ distinguishable balls are independently randomly numbered with 1,2 or 3, each number being equally likely. What is the probability that there are n balls with number 1, and n balls with number 2 and n balls with number 3.

2. Let $A_1, A_2, \ldots, A_m$ be subsets of A and $|A_i| = n$. Show that if $m < \frac{4^{n-1}}{3^n}$ then there is a way of coloring A with 4 colors so that each color appears at least once in each set.

3. Let $s_1, s_2, \ldots, s_m$ be binary strings such that no string is a prefix of another string.
   ($a = a_1 a_2 \cdots a_p$ is a prefix of $b = b_1 b_2 \cdots b_q$ if $p \leq q$ and $a_i = b_i$ for $1 \leq i \leq p$).
   Show that
   $$\sum_{i=1}^{m} 2^{-|s_i|} \leq 1$$
   where $|s|$ is the length of string s.
   (Hint: Let $n = \max\{|s_1| : 1 \leq i \leq n\}$. Let x be a random binary string of length n. Consider the events $\mathcal{E}_i = \{s_i \text{ is a prefix of } x\}$.)