21-301 Combinatorics
Homework 3
Due: Monday, September 19

1. A group of $2n$ people is seated around a round table. The group leaves the table for a break and then returns. In how many ways can the people sit down after returning so that no one is sitting directly opposite to the same person in both seatings? (If the positions round the table are $1, 2, \ldots, 2n$ then $n+i$ is directly opposite i for $1 \le i \le n$).

   Solution: Label the table positions as $1, 2, \ldots, 2n$ as well as the people. Let A_i, $1 \le i \le n$ be the set of placings of the people so that i and $n+i$ sit opposite to each other i.e. at j and $n+j$ for some j. We need to determine $|A_S|$ for $S \subseteq [n]$.

   Suppose $|S| = k$, then
   $$|A_S| = \binom{n}{k} k! 2^k (2n-2k)!.$$
   We choose k positions among table places $1, 2, \ldots, n$ where the k pairs are to go. We assign the pairs to the places in $k!$ ways and then for each pair there are 2 choices as to which of them goes in the lower numbered place. The remaining people can be placed in $(2n-2k)!$ ways.

   Thus, the number of placings is
   $$\sum_{k=0}^{n} \sum_{|S|=k} (-1)^k \binom{n}{k} k! 2^k (2n-2k)! = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 k! 2^k (2n-2k)!.$$

2. Show that the number of permutations of $[n]$ which do not contain a consecutive pair of the form $k, k+1$ satisfies the recurrence
   $$b_n = (n-1)b_{n-1} + (n-2)b_{n-2}.$$
   [Hint: delete n from such a sequence and separately count those permutations which still satisfy the condition and those that don't.]

   Solution: Let
   $$B_n = \{\pi : \nexists \; consecutive \; pair \; k, k+1, \; 1 \le k < n\}$$
   so that $b_n = |B_n|$.

   If we remove n from the sequence $\pi(1), \pi(2), \ldots, \pi(n)$ then we obtain a permutation $\hat{\pi}$ of $[n-1]$. Let
   $$B_n' = \{\pi \in B_n : \; \hat{\pi} \in B_{n-1}\} \; and \; B_n'' = B_n \setminus B_n'.$$
   Let
   $$B_{n,k}' = \{\pi \in B_n' : \; n \; follows \; k\}$$
   where $k = 0, 1, \ldots, n-2$. (n follows 0 means that it goes at the front).

   The map "place n after k" defines a bijection between B_{n-1} and $B_{n,k}'$ and so $|B_{n,k}'| = |B_{n-1}| = b_{n-1}$. (The inverse of this map is "remove n").

Next let

$$B''_{n,k} = \{\pi \in B''_n : \ n \text{ lies between } k \text{ and } k+1\}$$

where $k = 1, \ldots, n-2$.

If, in the sequence defined by π, we remove n and replace $k, k+1$ by k and $i > k$ by $i-1$ then we obtain a member of B_{n-2}. The inverse of this map is to replace $i > k$ by $i+1$ and then replace k by the sequence $k, n, k+1$. Thus, $|B''_{n,k}| = |B_{n-2}| = b_{n-2}$ and we get our recurrence.

3. Let $a_0, a_1, a_2, \ldots$ be the sequence defined by the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2} + 1 \quad \text{for } n \geq 2$$

with initial conditions $a_0 = 0$ and $a_1 = 1$. Determine the generating function for this sequence, and use the generating function to determine a_n for all n.

Solution:

$$\begin{aligned}
\sum_{n=2}^{\infty}(a_n - a_{n-1} - 2a_n)x^n &= -\sum_{n=2}^{\infty} x^n \\
a(x) - x - xa(x) - 2x^2a(x) &= -\frac{x^2}{1-x} \\
a(x)(1 - x - 2x^2) &= -x - \frac{x^2}{1-x}
\end{aligned}$$

$$\begin{aligned}
a(x) &= -\frac{x}{(1+x)(1-2x)} - \frac{x^2}{(1-x^2)(1-2x)} \\
&= -\frac{1/6}{1+x} - \frac{1/2}{1-x} + \frac{2/3}{1-2x} \\
&= \sum_{n=0}^{\infty}\left(-\frac{1}{6}(-1)^n - \frac{1}{2} + \frac{2}{3}2^n\right)x^n.
\end{aligned}$$

So

$$a_n = -\frac{1}{6}(-1)^n - \frac{1}{2} + \frac{2}{3}2^n \qquad \text{for } n \geq 0.$$