21-301 Combinatorics
Homework 1
Due: Wednesday, September 14

1. Use the binomial theorem to prove that

$$\binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \cdots = 2^{n/2}\cos n\pi/4.$$

$$\binom{n}{1} - \binom{n}{3} + \binom{n}{5} - \binom{n}{7} + \cdots = 2^{n/2}\sin n\pi/4.$$

[Hint: $\cos\theta + i\sin\theta = e^{i\theta}$ where i denotes $\sqrt{-1}$. Now consider the expansion of $(1+i)^n$.]

Solution: Using the binomial theorem,

$$\begin{aligned}(1+i)^n &= \binom{n}{0} + \binom{n}{1}i + \binom{n}{2}i^2 + \binom{n}{3}i^3 + \binom{n}{4}i^4 + \cdots + \\ &= \left(\binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \cdots\right) + i\left(\binom{n}{1} - \binom{n}{3} + \cdots\right)\end{aligned}$$

Thus

$$\begin{aligned}Re\ (1+i)^n &= \binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \cdots. \\ Im\ (1+i)^n &= \binom{n}{1} - \binom{n}{3} + \binom{n}{5} - \cdots.\end{aligned}$$

On the other hand

$$(1+i)^n = 2^{n/2}(\cos\pi/4 + i\sin\pi/4)^n = 2^{n/2}(\cos n\pi/4 + i\sin n\pi/4)$$

and so

$$\begin{aligned}Re\ (1+i)^n &= 2^{n/2}\cos n\pi/4 \\ Im\ (1+i)^n &= 2^{n/2}\sin n\pi/4\end{aligned}$$

2. Show that for a fixed k,

$$\sum_{\ell=0}^{n-k}\binom{n}{k,\ell,n-k-\ell} = 2^{n-k}\binom{n}{k}. \tag{1}$$

[Hint: Expand $(1+x+y)^n$ using the multinomial theorem. Then put $y=1$ and extract the coefficient of x^k in what remains.]

Solution: From the multinomial theorem

$$(1+x+y)^n = \sum_{k=0}^{n}\left(\sum_{\ell=0}^{n-k}\binom{n}{k,\ell,n-k-\ell}y^\ell\right)x^k.$$

Putting $y = 1$ we get

$$(2+x)^n = \sum_{k=0}^{n} \left(\sum_{\ell=0}^{n-k} \binom{n}{k, \ell, n-k-\ell} \right) x^k.$$

But

$$(2+x)^n = \sum_{k=0}^{n} 2^{n-k} \binom{n}{k} x^k$$

and so we obtain (1).

3. Of the 16! orderings of the letters $A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P$ how many are there such that we cannot obtain any of the words BAD, DEAF, APE by crossing out some letters?

   Solution: Let Ω be the set of all orderings of the letters A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P. Let A_1 be the set of strings in Ω such that we cannot obtain the word BAD by crossing out some of the letters. Let A_2 and A_3 be the corresponding sets for the words DEAF and APE, respectively. The set we would like to count is $\Omega \setminus (A_1 \cup A_2 \cup A_3)$. We apply inclusion/exclusion.

   Note that we have $|\Omega| = 16!$ and (why?)

$$|A_1| = \binom{16}{3} 13! \qquad |A_2| = \binom{16}{4} 12! \qquad |A_3| = \binom{16}{3} 13!.$$

   It remains to compute the cardinalities of all possible intersections of the sets A_1, A_2 and A_3.

   We begin with $A_1 \cap A_2$. Let x be a string in $A_1 \cap A_2$. Since x is in A_1, we can cross out some set of letters and get the word BAD; in other words, B comes before A which comes before D in x. In particular, A comes before D in x. Since x is in A_2 we can get the word DEAF from x by crossing out some letters. This implies, in particular, that D comes before A in x. This is a contradiction. Therefore $A_1 \cap A_2 = \emptyset$.

   A similar argument shows that $A_2 \cap A_3 = \emptyset$.

   Now consider $A_1 \cap A_3$. This set is nonempty as there is no immediate contradiction arising from the words BAD and APE. We claim that once the positions of the letters A,B,D,E and P are determined there are exactly 3 ways to assign these five letters to the specified 5 positions which respects the condition that the string is in $A_1 \cap A_3$. (B must be first, followed by A. Since P must come before E there are 3 ways to fill in the remaining positions.) Therefore.

$$|A_1 \cap A_3| = \binom{16}{5} \cdot 3 \cdot 11!.$$

   Noting that $A_1 \cap A_2 = \emptyset$ implies $A_1 \cap A_2 \cap A_3 = \emptyset$, inclusion/exclusion gives

$$\begin{aligned} \Omega \setminus (A_1 \cup A_2 \cup A_3) &= \sum_{S \subseteq \{1,2,3\}} (-1)^| S|A - S \\ &= |\Omega| - |A_1| - |A_2| - |A_3| + |A_1 \cap A_2| + \\ &\qquad |A_1 \cap A_3| + |A_2 \cap A_3| - |A_1 \cap A_2 \cap A_3| \\ &= 16! - \binom{16}{3} 13! - \binom{16}{4} 12! - \binom{16}{3} 13! + \binom{16}{5} \cdot 3 \cdot 11! \end{aligned}$$