21-301 Combinatorics
Homework 1: Solutions
Due: Wednesday, September 7

1. How many integral solutions of

$$x_1 + x_2 + x_3 + x_4 = 30$$

satisfy $x_1 \geq 2$, $x_2 \geq 0$, $x_3 \geq -5$ and $x_4 \geq 8$?

Solution Let

$$y_1 = x_1 - 2, \qquad y_2 = x_2, \qquad y_3 = x_3 + 5, \qquad y_4 = x_4 - 8.$$

An integral solution of $x_1 + x_2 + x_3 + x_4 = 30$ such that $x_1 \geq 2$, $x_2 \geq 0$, $x_3 \geq -5$ and $x_4 \geq 8$ corresponds to an integral solution of $y_1 + y_2 + y_3 + y_4 = 25$ such that $y_1, \ldots, y_4 \geq 0$. From a result in class,

$$|\{(y_1, y_2, y_3, y_4) : y_1, \ldots, y_4 \in Z_+ \text{ and } y_1 + \cdots + y_4 = 25\}| = \binom{25 + 4 - 1}{4 - 1} = \binom{28}{3}.$$

2. Prove the following equality using a *combinatorial* argument

$$\sum_{i=1}^{n} i \binom{n}{i} = n2^{n-1}.$$

Solution Consider the set

$$\mathcal{S} = \{(A, x) : A \subseteq [n] \quad \text{and} \quad x \in A\}.$$

In words, $\mathcal{S}$ is the set of all ordered pairs consisting of a subset of $[n]$ and some element of that set. We count the elements of $\mathcal{S}$ is two ways.

First we count with respect to the element x. There are n choices for x. Once x is fixed, $A \setminus \{x\}$ can be any subset of $[n] \setminus \{x\}$. There are 2^{n-1} such sets. Therefore, we have

$$|\mathcal{S}| = n2^{n-1}.$$

Now we count with respect to the set A. For $i = 1, 2, \ldots, n$ let

$$\mathcal{S}_i = \{(A, x) \in \mathcal{S} : |A| = i\}.$$

These sets form a partition of $\mathcal{S}$. There are $\binom{n}{i}$ choices for the set A in an ordered pair in $\mathcal{S}_i$. Once this set is fixed there are i choices for x. Therefore

$$|\mathcal{S}_i| = \binom{n}{i} i,$$

and

$$|\mathcal{S}| = \sum_{i=1}^{n} |\mathcal{S}_i| = \sum_{i=1}^{n} \binom{n}{i} i.$$

The result is given by noting that the two expressions for $|\mathcal{S}|$ are equal.

3. How many functions $f : [n] \to [n]$ are there which satisfy $i < j$ implies $f(i) \leq f(j)$?

Solution Let $x_j = f(j+1) - f(j)$ for $j = 1, \ldots, n-1$ and let $x_0 = f(1) - 1$ and $x_{n+1} = n$. Then

$$x_j \geq 0 \text{ for } j = 0, 1, \ldots, n \text{ and } x_0 + x_1 + \cdots + x_n = n. \tag{1}$$

Conversely, given a sequence satisfying (1) we get a monotone increasing function by defining $f(j) = 1 + x_0 + \cdots + x_{j-1}$ and the mappings f to x are inverse to each other. Thus the number of f's is equal to the number of x's satisfying (1), which is $\binom{2n-1}{n}$.