21-301 Combinatorics Homework 10 Due: Monday, December 5

1. Consider the following game: There is a pile of n chips. A move consists of removing any *proper* factor of n chips from the pile. The player to leave a pile with one chip wins. Determine the N and P positions and a winning strategy from an N position.

Solution n is a P-position iff it is odd. If n is even then the next player can simply remove one chip. If n is odd, then any factor f of n is also odd and then n - f is even.

2. Two players play the following game on a round table top of radius R. Players take turns placing pennies (of unit radius) on the tabletop, but no penny is allowed to overlap. Which of the players has a winning strategy as a function of R and why?

Solution Player one's strategy is to place his/her first coin in the middle of the table and then place the center of any subsequent coin at (-x, -y), assuming that Player 2 has just placed a coin at position (x, y). This will be possible, since the placement of the pennies remains symmetric about the origin after each of the first players moves.

3. Consider the following game: There is a single pile of n chips. A move consists of removing (i) any number of chips divisible by three provided it is not the whole pile, or (ii) the whole pile, but only if it has 2 (mod 3) chips. The terminal positions are zero, one and three. Determine the Sprague-Grundy number f(n) of a pile size n. (Hint: Compute the first few numbers and look for a pattern)

Solution We have

$$f(n) = \begin{cases} 0 & n = 0, 1, 3\\ 1 & n = 2\\ a + b - 1 & 3 \le n = 3a + b \end{cases}$$

Observe first that f(0) = f(1) = f(3) = 0 since 0,1,3 are terminal positions. $f(2) = mex\{f(0)\} = mex\{0\} = 1$. For the rest we will use induction on n and our base case is $n = 3 = 3 \times 1 + 0$.

So suppose that n = 3a + b where $a \ge 1$ and b = 0, 1, 2 and that f(n') is as described for n' < n. Then

$$f(3a) = mex\{f(3a-3), f(3a-6), \dots, f(3)\}\$$

= $mex\{a-2, a-3, \dots, 0\} = a-1.$

$$f(3a+1) = mex\{f(3a-2), f(3a-5), \dots, f(1)\}\$$

= mex{a-1.a-2, ..., 0} = a-2.

$$f(3a+2) = mex\{f(3a-1), f(3a-4), \dots, f(2), f(0)\}\$$

= mex{a-1.a-2,...,0} = a-1.

And this completes our induction.