Department of Mathematics
Carnegie Mellon University

21-301 Combinatorics, Fall 2005: Test 4

Name:________________________________

Problem	Points	Score
1	33	
2	33	
3	34	
4	34	
Total	100	

Q1: (33pts)
Consider the following general game involving one pile of chips . There is a finite set of positive integers S and each move involves choosing $s \in S$ and removing this number of chips from the pile. The game ends when there are no moves possible. Show that the SPRAGUE-GRUNDY function g satisfies $g(n) \leq |S|$ for all $n \geq 0$.

Q2: (33pts)
Consider the following game involving two piles of chips. A move consists of removing one pile and splitting the remaining pile into two non-empty piles. There is a unique terminal position in which both piles have one chip.
Suppose that the two piles have m, n chips respectively. Here are the N, P positions for $1 \leq m, n \leq 6$.

	1	2	3	4	5	6
1	P	N	P	N	P	N
2	N	N	N	N	N	N
3	P	N	P	N	P	N
4	N	N	N	N	N	N
5	P	N	P	N	P	N
6	N	N	N	N	N	N

When in general is m, n a P position. Prove your claim.

Q3: (34pts)
Consider the following game involving one pile of chips. A move consists of removing 2^k chips where $k = 0, 1, 2, 3, \ldots$. The first few values of the SPRAGUE-GRUNDY function g are given in the following table:

n	0	1	2	3	4	5	6	7	8	9	19	11	12	13	14	15	16
$g(n)$	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1

What is $g(n)$ in general? Prove your claim by induction.

Q4: (33pts)

An $m \times n$ 0,1 matrix A and a $p \times q$ 0,1 matrix B are *compatible* if $A(i,j) = B(i,j)$ for $1 \le i \le \min\{m,p\}$ and $1 \le j \le \min\{n,q\}$.

Suppose that A_i is an $m_i \times n_i$ $0,1$ matrix for $i = 1, 2, \ldots, N$ and that there do **not** exist i, j such that A_i is compatible with A_j. Prove that

$$\sum_{i=1}^{N} \frac{1}{2^{m_i n_i}} \le 1.$$