Combinatorial Analysis 21-301: Fall 2003
Homework.
HW9 due Friday 11/7/2003

Q1: Let A be a 0-1 $m \times n$ matrix. If $S \subseteq [n]$ then A_S is the $m \times |S|$ submatrix whose columns are the columns $A_i, i \in S$. A is said to be k-universal if every set S of k columns has the following property: Every vector in $\{0,1\}^k$ appears as a row of A_S.
Show that if $\binom{n}{k} 2^k \left(1 - \frac{1}{2^k}\right)^m < 1$ then there exists at least one k-universal matrix.
Solution For $S \subseteq [n], |S| = k$ and $x \in \{0,1\}^k$ let $\mathcal{E}_{S,x}$ be the event that the matrix A_S does not contain a row equal to x. Then

$$\begin{aligned}
\mathbf{Pr}(\neg\text{property}) &= \mathbf{Pr}\left(\bigcup_{S,x} \mathcal{E}_{S,x}\right) \\
&\leq \sum_{S,x} \mathbf{Pr}(\mathcal{E}_{S,x}) \\
&= \sum_{S,x} \left(1 - \frac{1}{2^k}\right)^m \\
&= \binom{n}{k} 2^k \left(1 - \frac{1}{2^k}\right)^m \\
&< 1.
\end{aligned}$$

Q2: Let $p = (1+\epsilon)\frac{\log n}{n}$ where $\epsilon > 0$ is constant. Show that **whp** $G_{n,p}$ is 2-connected. (A graph is k-connected if removing any $k-1$ or less vertices leaves it connected.)
Solution We already know from class that $G_{n,p}$ is connected **whp**. For $S \subseteq [n], 1 \leq |S| \leq n/2$ and $v \in [n] \setminus S$ let $\mathcal{E}_{S,v}$ be the event that S is adjacent

to v and S is a component of $G_{n,p} - v$. Then

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\begin{aligned}
&\mathbf{Pr}(G_{n,p} \text{ is connected and not 2-connected}) \\
&= \mathbf{Pr}\left(\bigcup_{S,v} \mathcal{E}_{S,v}\right) \\
&\leq \sum_{S,v} \mathbf{Pr}(\mathcal{E}_{S,v}) \\
&\leq n \sum_{k=1}^{n/2} \binom{n}{k} k^{k-2} p^{k-1} kp(1-p)^{k(n-k-1)} \\
&\leq n \sum_{k=1}^{n/2} \left(nepe^{-(n-k-1)p}\right)^k \\
&\leq n\left(\frac{(1+\epsilon+o(1))e\log n}{n^{1+\epsilon}}\right) + n \sum_{k=2}^{n/2} \left(\frac{(1+\epsilon+o(1))e\log n}{n^{(1+\epsilon)/2}}\right)^k \\
&= o(1).
\end{aligned}
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