Combinatorial Analysis 21-301: Fall 2003
Homework.
HW8 due Monday 11/3/2003

Q1: A necklace is made of 6 beads strung together in a cycle. Compute the pattern inventory for colouring the beads using two colours if the group G is the **dihedral group** generated by rotations about the centre plus flips about a diameter. Using the notation R to represent a clockwise rotation of $\pi/3$ and F to represent a flip on the diameter 1–4, the group G consists of the 12 elements $\{R^iF^j : 0 \leq i \leq 5, j = 1, 2\}$.
Here we can reduce any sequence of rotations and flips to the 12 given by using the relations $R^6 = F^2 = e$ (identity) and $FR = R^5F$.
See http://merganser.math.gvsu.edu/david/reed03/projects/ettingerGuy/ for a nice discussion.
Solution:

g	e	R	R^2	R^3	R^4	R^5	F	RF	R^2F	R^3F	R^4F	R^5F
$ct(g)$	x_1^6	x_6	x_3^2	x_2^3	x_3^2	x_6	$x_1^2x_2^2$	x_2^3	$x_1^2x_2^2$	x_2^3	$x_1^2x_2^2$	x_2^3

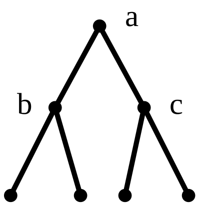
Thus the cycle index polynomial is

$$\frac{1}{12}(x_1^6 + 3x_1^2x_2^2 + 4x_2^3 + 2x_3^2 + 2x_6)$$

and the pattern inventory is

$$\frac{1}{12}((B+W)^6+3(B+W)^2(B^2+W^2)^2+4(B^2+W^2)^3+2(B^3+W^3)^2+2(B^6+W^6)).$$

Q2: Consider a complete binary tree on 7 nodes (drawn below): Compute the pattern inventory for colouring the edges using two colours if the group G can rotate the tree below each vertex. This group has 8 elements: Let e_a, e_b, e_c denote rotations under vertices a, b, c respectively. Then dropping the $\circ$ we have $e_ae_b = e_ce_a$, $e_ae_c = e_be_a$ and $e_be_c = e_ce_b$. So G consists of $\{e, e_a, e_b, e_c, e_ae_b, e_ae_c, e_be_c, e_ae_be_c\}$.



Solution:

g	e	e_a	e_b	e_c	e_ae_b	e_ae_c	e_be_c	$e_ae_be_c$
$ct(g)$	x_1^6	x_2^3	$x_1^4x_2$	$x_1^4x_2$	x_2x_4	x_2x_4	$x_1^2x_2^2$	x_2^3

Thus the cycle index polynomial is

$$\frac{1}{8}(x_1^6 + x_1^2x_2^2 + 2x_2^3 + 2x_1^4x_2 + 2x_2x_4)$$

and the pattern inventory is

$$\frac{1}{8}((B+W)^6 + (B+W)^2(B^2+W^2)^2 + 2(B^2+W^2)^3 \\ + 2(B+W)^4(B^2+W^2) + 2(B^2+W^2)(B^4+W^4)).$$